Egzaminuojame Moderniųjų technologijų studentą o1-p

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21/01/2025, Vilnius

Šiame semestre daug diskusijų vyko apie mūsų naujus "studentus", vieno iš jų "o1-p" rezultatus ir aptarsime.

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Klausimus parinko tikrai kvalifikuotas ir labai populiarus kolega-matematikas Dr. Alon Amit (Vice President, Product Intuit).

Užduotys ne pačios lengviausios, bet tikrai įveikiamos studentams, kurie sąžiningai mokėsi visą semestrą (tikiuosi, kad ir mūsų dėstytojai sugebės šias užduotis išspręsti).

1. Prove that among 501 integers between 1 and 1000 there must be two such that one divides the other.

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Mano lūkesčiai:

Tikėjausi, kad studentas pademonstruos savo matematinę brandą ir atsakymo pradžioje suformuluos paprastą bet svarbų matematinėje diskusijoje teiginį, kad 500 skaičių neužteks, bei pateiks kokį nors nesudėtingą ir elegantišką kontrapavyzdį.

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In mathematics, the Pigeonhole principle states that if n items are put into m containers, with $n > m$, then at least one container must contain more than one item.

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Since there are only 31 possible values for the largest divisor $(1, 2, 3, \ldots, 31$, as the square root of 1000 is approximately 31.62), and we have 501 integers, by the Pigeonhole Principle, at least two integers must have the same largest divisor.

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Pvz.: $a = 9, b = 15$, $\sqrt{a} = 3$, $\sqrt{15} < 4$, tai bendras didžiausias daliklis yra 3.

Let's call these two integers a and b. Without loss of generality, assume $a < b$.

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Now, if $b - a$ is zero , then $a = b$, which means a divides b.

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Kaip tai gali būti, jeigu a ir b yra skirtingi?

If $b - a$ is non-zero, then d divides $b - a$, which means $a + (b - a) = b$ is divisible by d, implying a divides b.

Thus, we have shown that among 501 integers between 1 and 1000, there must be two such that one divides the other.

Tai klaidingas teiginys, imkite kontrapavyzdį: mūsų jau nagrinėtas skaičius 15 nesidalina iš 9.

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Įdomus studento bandymas pateikti "moksliškesnį" atsakymą, kai bendrų daliklių paieška buvo siejama su dalikliais, nedidesniais už kvadratinę skaičių šaknį. Tai teisingas žingsnis, kai tikriname, ar skaičius yra pirminis, bet...

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Insanity is doing the same thing over and over and expecting different results (Albert Einstein).

Let's think about it: we have 501 integers between 1 and 1000.

We can group these integers into sets based on their prime factors.

For example, the numbers $2, 4, 6, 8, \dots$ all have 2 as a prime factor, so they can be grouped together.

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Skaičius $21 = 3 \cdot 7$, kuriai grupei jis priklauso? Turime dvi grupes, ar tai nesugadins įrodymo?

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Čia neteisingai naudojamas Pigeonhole pricipas, pirminių skaičių yra 168, tarkime, kad tiek bus ir grupių.

Let's say we have two integers, a and b, in the same group. This means they share a common prime factor, p. Without loss of generality, let's assume $a < b$.

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Now, since a and b share a common prime factor p, we can write $a = p^k \cdot m$ and $b = p^l \cdot n$, where k, l, m, and n are integers, and m and n are not divisible by p.

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Tai klaidinga išvada. Imkime $a = 9$, $b = 15$. Tada

$$9 = 3^2,\ k = 2,\ \ 15 = 3 \cdot 5,\ l = 1$$

ir $k > l$.

If $k = l$, then a divides b (because m and n are relatively prime to p).

If $k < l$, then a still divides b, because $p^{(l-k)} \cdot m$ divides $p^{l} \cdot n$.
So, we've shown that among the 501 integers, there must be two such that one divides the other.

Teisingas įrodymas (tai vienas iš galimų).

Apsiribosime skaičiais $\{1, \ldots, 100\}$ ir išrenkame atsitiktinai 51 skaičių.

Kiekvienas iš jų gali būti užrašytas taip:

$$x = 2^m n, \quad n \in A = \{1, 3, ..., 99\}.$$

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Matome, kad $|A| = 50$ (vienu elementu mažiau, nei 51, tiek elementų pasirinkome visoje aibėje).

Taigi, remiantis Pigeonhole Principle, turi egzistuoti bent du skaičiai, kuriems n sutampa

$$x = 2^m n, \quad y = 2^k n, \quad k > m.$$

Tada

$$y/x = 2^k n/(2^m n) = 2^{k-m} > 1.$$

Task 2.

Let V be a finite dimensional vector space over a field, and U, W two subspaces. Is it true that the dimension of the sum of U and W is equal to the sum of their dimensions minus the dimension of their intersection?

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A great question from linear algebra!

The answer is yes, it is true that the dimension of the sum of U and W is equal to the sum of their dimensions minus the dimension of their intersection.

This is a well-known result in linear algebra, often referred to as the "Dimension Formula" or "Grassmann's Formula". It can be stated as:

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Here, $U + W$ denotes the sum of the subspaces U and W, and $U \cap W$ denotes their intersection.

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$$- \dim(U \cap V) - \dim(U \cap W) - \dim(V \cap W) + \dim(U \cap V \cap W).$$

This formula is known as the "Inclusion-Exclusion Principle" or "Principle of Inclusion-Exclusion" in linear algebra.

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In combinatorics, the Inclusion–Exclusion principle is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets.

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Wait that isn't true.

I apologize for the mistake.

The formula I provided earlier is not correct for three subspaces.

The correct generalization of the dimension formula for three subspaces U, V, and W is more complex and involves using the concept of intersection and sum of subspaces.

If you're interested, I can try to provide a correct formula or a resources that explains it. Alternatively, if you'd like to discuss the correct approach or provide more context, I'm here to help!

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Studentas pataisė savo atsakymą ir pridėjo papildomą narį

$$\dim\big((U \cap V) + (U \cap W) + (V \cap W)\big).$$

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Imkime seką neneigiamų skaičių $X = \{x_1, x_2, x_3, \ldots\}$, kai $x_n \geq 0$.

Sudarykime dalinių sumų seką

$$s_n = x_1 + \ldots + x_n, \quad n = 1, 2, \ldots$$

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Ar seka S turi ribą?

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So, to answer your question, the sequence does not necessarily converge just because the differences tend to 0.

Task 4.

Find all possible arrangements of four distinct points in a plane such that only two distances occur between them.

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Here are two solutions to the problem of configurations of 4 points in the plane with only 2 distinct distances:

Solution 1: Square Configuration

One possible configuration is when the four points form a square. In this case, all adjacent points are at the same distance (let's call it d), and all diagonal points are at the same distance sqrt(2)d.

Solution 2: Equilateral Triangle with a Point at the Centroid

Another possible configuration is when three points form an equilateral triangle, and the fourth point is at the centroid (the point of intersection of the medians) of the triangle.

Let's call the side length of the equilateral triangle d. Then, the distance between the centroid and each vertex is d/ sqrt(3). Therefore, we have two distinct distances: d (between adjacent vertices of the triangle) and d sqrt(3) (between the centroid and each vertex).

Klausimas: what about Rhombus?

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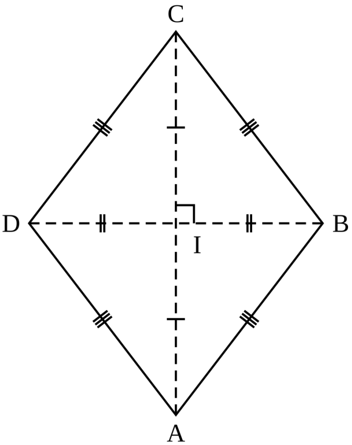
A rhombus is indeed another configuration of 4 points in the plane with only 2 distinct distances.

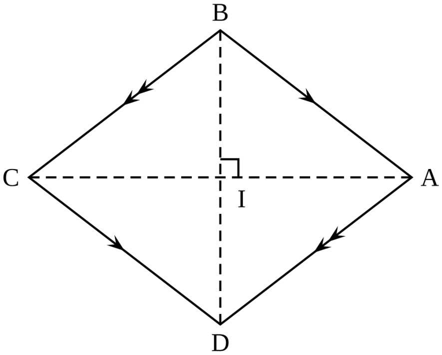
In a rhombus, all sides have the same length (let's call it d), and all diagonals have the same length. Therefore, we have two distinct distances: d (between adjacent points) and d_2 (between opposite points).

So, we can add the rhombus configuration to our list of solutions.

Thank you for pointing out the rhombus solution!

Pastaba: Yes, but the lengths of two diagonals are not necessarily equal?





I apologize for the mistake. In a rhombus, the two diagonals are not necessarily equal in length.

Correction: Rhombus

Points: A, B, C, D
Distances: $AB = BC = CD = DA = 1$ (short distance)

Distances: AC and BD (no restriction on diagonal lengths)

In a rhombus, all four sides have the same length (short distance), but the lengths of the two diagonals can be different. This means that a rhombus does not necessarily have two distinct distances, as the diagonals can have different lengths.

Thank you for pointing out my mistake! I'll make sure to be more accurate in the future.