Multiscale modelling of viscous flows in domains of complex geometry

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Plan

- introduction;
- time-periodic Stokes equations in a domain with an outlet to infinity;
- time-periodic Navier-Stokes equations in a thin tube structure;
- steady-state Navier-Stokes equations with the Bernoulli pressure in a thin tube structure;
- conclusion.

Navier-Stokes equations

$$
\begin{cases}
\mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f}, & (x,t) \in \Omega \times (0,2\pi), \\
\operatorname{div}\mathbf{v} = 0, & (x,t) \in \Omega \times (0,2\pi), \\
\mathbf{v}(x,t) = \boldsymbol{\varphi}(x,t), & (x,t) \in \partial\Omega \times (0,2\pi), \\
\mathbf{v}(x,0) = \mathbf{v}(x,2\pi), & x \in \Omega,
\end{cases}
$$

- $\mathbf{v} = \mathbf{v}(x,t) = (v_1(x,t),\ldots,v_n(x,t))$ - unknown velocity;
- $p = p(x,t)$ - unknown pressure;
- $\mathbf{f} = \mathbf{f}(x,t) = (f_1(x,t),\ldots,f_n(x,t))$ - given external force;
- $\boldsymbol{\varphi} = \boldsymbol{\varphi}(x,t) = (\varphi_1(x,t),\ldots,\varphi_n(x,t))$ - given boundary value;
- $\mathbf{v}(x,0) = \mathbf{v}(x,2\pi)$ - periodicity condition;
- $x = (x_1,\ldots,x_n) \in \Omega$;
- $\nu > 0$ - the viscosity coefficient.

Time-periodic Stokes system

We consider the time-periodic Stokes system in a domain Ω

$$\begin{cases} \mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, & (x,t) \in \Omega \times (0, 2\pi), \\ \operatorname{div} \mathbf{v} = 0, & (x,t) \in \Omega \times (0, 2\pi), \\ \mathbf{v}(x,t) = \boldsymbol{\varphi}(x), & (x,t) \in \partial\Omega \times (0, 2\pi), \\ \mathbf{v}(x,0) = \mathbf{v}(x, 2\pi), & x \in \Omega. \end{cases} \tag{1}$$

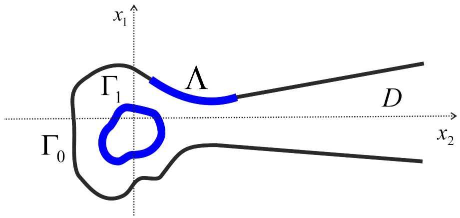


Figure: Domain Ω

Domain Ω with an outlet to infinity

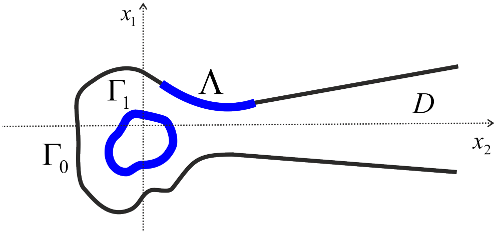


Figure: Domain Ω

An outlet to infinity $D = \{x \in \mathbb{R}^2 : |x_1| < g(x_2), x_2 > R_0\}$.
We suppose that:

- the function g satisfies the Lipschitz condition

$$|g(t_1) - g(t_2)| = L|t_1 - t_2|, t_1, t_2 > R_0, g(t) \geq \text{const} > 0;$$

- boundary value $\varphi \in W^{3/2,2}(\partial\Omega)$ has a compact support.

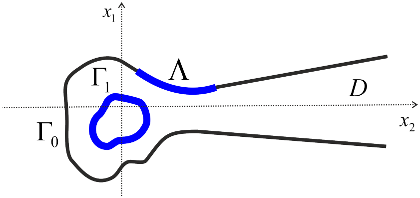
Time-periodic Stokes equations

Definition

By weak solution of problem (1) we understand a solenoidal vector field $\mathbf{v}$ with $\nabla\mathbf{v}, \mathbf{v}_t \in L^2(0, 2\pi; L^2(\Omega))$ satisfying the boundary condition $\mathbf{v}|_{\partial\Omega} = \boldsymbol{\varphi}$, the time periodicity condition $\mathbf{v}(x, 0) = \mathbf{v}(x, 2\pi)$ and the integral identity

$$\int\limits_0^{2\pi}\int\limits_\Omega \mathbf{v}_t \cdot \boldsymbol{\eta}\,\mathrm{d}x\mathrm{d}t + \nu\int\limits_0^{2\pi}\int\limits_\Omega \nabla\mathbf{v} : \nabla\boldsymbol{\eta}\,\mathrm{d}x\mathrm{d}t = \int\limits_0^{2\pi}\int\limits_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\,\mathrm{d}x\mathrm{d}t$$

for all time-periodic $\boldsymbol{\eta} \in L^2(0, 2\pi; L^2(\Omega))$.



Since $\operatorname{div} \mathbf{v} = 0$, the necessary compatibility condition

$$\int\limits_{\sigma(R)} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}S = -\Big(\int\limits_{\Gamma_1} \boldsymbol{\varphi} \cdot \mathbf{n} \, \mathrm{d}S + \int\limits_{\Lambda} \boldsymbol{\varphi} \cdot \mathbf{n} \, \mathrm{d}S \Big) = -\big(\mathcal{F}^{(\mathsf{inn})} + \mathcal{F}^{(\mathsf{out})} \big)$$

holds. Here $\mathcal{F}^{(\mathsf{inn})}$ and $\mathcal{F}^{(\mathsf{out})}$ are the fluxes of the boundary value $\boldsymbol{\varphi}$ over the inner and the outer boundaries, respectively.

Boundary value extension

Boundary value $\mathbf{A}$ extension could be constructed using similar ideas as in paper written by K. Kaulakytė, K. Pileckas. [1]

We construct the extension $\mathbf{A}$ in the following form:

$$\mathbf{A}(x) = \mathbf{B}^{(\mathrm{inn})}(x) + \mathbf{B}^{(\mathrm{out})}(x),$$

where $\mathbf{B}^{(\mathrm{inn})}$ extends the boundary value φ from the inner boundary Γ_1, and $\mathbf{B}^{(\mathrm{out})}$ extends φ from the outer boundary Γ_0.

[1] K. Kaulakytė, K. Pileckas, On the nonhomogeneous boundary value problem for the Navier–Stokes system in a class of unbounded domains, J. Math. Fluid Mech. 14(4), 693–716 (2012).

Solvability of the time-periodic Stokes problem

We look for the solution of problem (1) in the form

$$\mathbf{v}(x,t) = \mathbf{A}(x) + \mathbf{u}(x,t),$$

where $\mathbf{A}$ is the suitable extension of the boundary value $\boldsymbol{\varphi}$. Let us denote the following space:

$$L^2_{\mathrm{per}}(0,2\pi; L^2_1(\Omega)) := \overline{C^\infty_{\mathrm{per}}(0,2\pi; L^2_1(\Omega))}^{L^2(0,2\pi)},$$

where $L^2_1(\Omega)$ is weighted space with the norm

$$\|w\|_{L^2_1(\Omega)} = \sqrt{\int\limits_D |w|^2 g^2 \, \mathrm{d}x + \int\limits_{\Omega_0} |w|^2 \, \mathrm{d}x}.$$

Solvability of the time-periodic Stokes problem

Theorem

Assume that the domain $\Omega \subset \mathbb{R}^2$ has one outlet to infinity, boundary value $\boldsymbol{\varphi} \in W^{3/2,2}(\partial\Omega)$ has a compact support, $\mathbf{f} \in L^2_{\mathrm{per}}(0, 2\pi; L^2_1(\Omega))$. If $\int\limits_1^{+\infty} \frac{\mathrm{d}x_2}{g^3(x_2)} < +\infty$, then problem (1) has a unique weak solution $\mathbf{v} = \mathbf{A} + \mathbf{u}$ satisfying the following estimate:

$$\|\mathbf{v}_t\|_{L^2(0,2\pi;L^2(\Omega))} + \|\nabla \mathbf{v}\|_{L^2(0,2\pi;L^2(\Omega))}$$
$$\leq c\Bigg(\bigg(\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)}\Big(1 + \int\limits_0^{+\infty} \frac{1}{g^3(x_2)}\,\mathrm{d}x_2\Big)\bigg)^{1/2} + \|\mathbf{f}\|_{L^2(0,2\pi;L^2_1(\Omega))}\Bigg).$$

These results were generalized by K. Kaulakytė and K. Pileckas,[2] when the boundary condition also depends on time, i.e. $\boldsymbol{\varphi} = \boldsymbol{\varphi}(x, t)$ and the Dirichlet integral $\int\limits_1^{+\infty} \frac{\mathrm{d}x_2}{g^3(x_3)}$ may be finite or infinite.

[2] K. Kaulakytė, K. Pileckas, Nonhomogeneous boundary value problem for the time periodic linearized Navier-Stokes system in a domain with outlet to infinity, Journal of Mathematical Analysis and Applications, 2020, https://doi.org/10.1016/j.jmaa.2020.124126

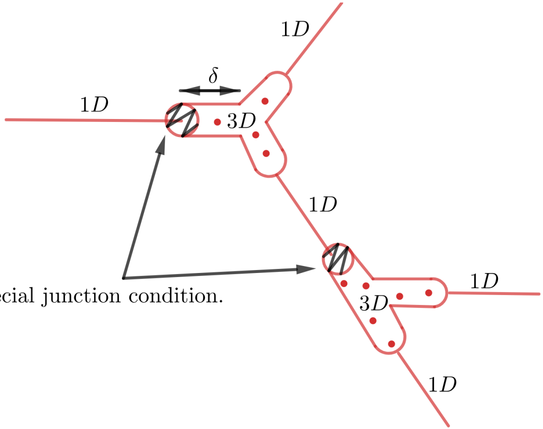
Motivation

The research was done participating in research project: 2017 - 2021 Junior research fellow of the research grant "Multiscale Modeling for Viscous Flows in Domains with Complex Geometry".[3] Our goal is to study Navier–Stokes equations in a tube structure. The main steps are:

- to introduce a tube structure B_ε, which describe some simplified blood vessel network and to consider the Navier–Stokes equations in it;
- to use small parameter ε. It is equal the ratio of the diameter of vessels to their length. This parameter generate two different scales;
- to formulate theorems about existence and uniqueness of the weak solution;
- to construct asymptotic expansions of the solution, which let us to combine hybrid dimensions.

[3]This project has received funding from European Social Fund (project No. 09.3.3-LMT-K-712-01-0012) under grant agreement with the Research Council of Lithuania (LMTLT).

Motivation



Special junction condition.

Graph of the tube structure

Denote $\mathcal{B} = \bigcup_{j=1}^{M} e_j$ the union of edges and assume that $\mathcal{B}$ is a connected set. The union of all edges having the same end point O_l is called the bundle $\mathcal{B}_l$.

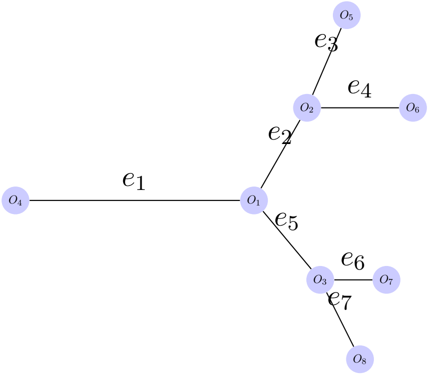


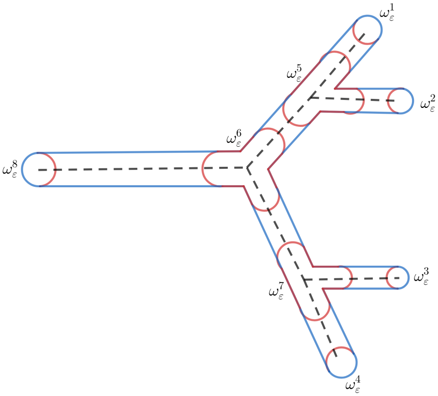
Figure: Graph of the tube structure

Definition of a graph

Definition

Let $O_1, O_2, \ldots, O_N$ be N different points in $\mathbb{R}^n, n = 2, 3$, and $e_1, e_2, \ldots, e_M$ be M closed segments each connecting two of these points (i.e. each $e_j = \overline{O_{i_j} O_{k_j}}$, where $i_j, k_j \in \{1, \ldots, N\}, i_j \neq k_j$). All points O_i are supposed to be the ends of some segments e_j. The segments e_j are called edges of the graph. A point O_i is called a node, if it is the common end of at least two edges and O_i is called a vertex, if it is the end of the only one edge. Any two edges e_j and e_i can intersect only at the common node. The set of vertices is supposed to be non-empty.

Definition of a tube structure



$\gamma_\varepsilon^j = \partial\omega_\varepsilon^j \cap \partial B_\varepsilon,$
$j = N_1 + 1, ..., N : O_j$ vertices.

Definition of the tube structure

Definition

By a tube structure, we call the following domain

$$B_\varepsilon = \Big(\bigcup_{j=1}^{M} \Pi_\varepsilon^{(e_j)}\Big) \bigcup \Big(\bigcup_{j=1}^{N_1} \omega_\varepsilon^j\Big).$$

Here we denote by $\Pi_\varepsilon^{(e)}$ the cylinder

$$\Pi_\varepsilon^{(e)} = \Big\{x^{(e)} \in \mathbb{R}^n : x_n^{(e)} \in (0, |e|), \frac{x^{(e)\prime}}{\varepsilon} \in \sigma^{(e)}\Big\},$$

and $\omega_\varepsilon^j = \{x \in \mathbb{R}^n : \dfrac{x - O_j}{\varepsilon} \in \omega^j\}$ the nodal domains. Suppose that it is a connected set and that the boundary ∂B_ε of B_ε is C^2-smooth.

Time-periodic Navier–Stokes equations in a thin tube structure

Consider the time-periodic boundary value problem for the Navier–Stokes equations in the tube structure B_ε

$$\begin{cases} \dfrac{1}{\varepsilon^\beta}\mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f}, \quad \beta = 0, 2, \\ \operatorname{div}\mathbf{v} = 0, \\ \mathbf{v}|_{\partial B_\varepsilon} = \mathbf{g}, \\ \mathbf{v}(x,t) = \mathbf{v}(x, t+2\pi), \end{cases} \tag{2}$$

here the fluid velocity $\mathbf{g}$ at the boundary ∂B_ε has the following structure: $\mathbf{g} = 0$ everywhere on ∂B_ε except for the set $\gamma_\varepsilon^{N_1+1}, \ldots, \gamma_\varepsilon^N$ where $\gamma_\varepsilon^j = \partial B_\varepsilon \cap \partial\omega_\varepsilon^j$, $j = N_1+1, \ldots, N$. And ε is a small parameter equal to the ratio of the diameter of vessels to their length.

Different scalings of the Navier–Stokes equations

We consider the problem in two different scalings concerning the small parameter ε: one of them is ε^0 (the same as proposed by G. Panasenko and K. Pileckas[4]), while the other generates a big coefficient of order ε^{-2} of the time derivative of the velocity. These scalings satisfy different types of vessels such as small and very small arterioles or capillaries.

Scalings were justified by using averaged data for E. N. Marieb and K. Hoehn book [5] and H. N. Mayrovitz paper [6].

[4] G. Panasenko, K. Pileckas, Asymptotic analysis of the non-steady Navier–Stokes equations in a tube structure. I. The case without boundary-layer-in-time, Nonlinear Anal. Theory, Methods Appl. 122, 125-168 (2015), Asymptotic analysis of the non-steady Navier–Stokes equations in a tube structure. II. General case, Nonlinear Anal. Theory, Methods and Appl. 125, 582-607 (2015).

[5] E. N. Marieb, K. Hoehn, Human Anatomy and Psysiology. The Cardiovascular System: Blood Vessels, Pearson, Boston, 9th ed., 2013, p. 712

[6] H. N. Mayrovitz, Skin capillary metrics and hemodynamics in the hairless mouse, Microvasc. Res. 43(1), 46–59 (1992)

Different scalings of the Navier–Stokes equations

Consider two different scalings where characteristic time is 1 second and the characteristic velocity is about $0.5 \times 10^{-3}\ m/sec$.

- The characteristic length is $10^{-3}\ m$;
  the characteristic diameter is $10^{-4}\ m$ ($\varepsilon = 0.1$);
  change of space variable $X = 10^{-3} x$;
  change of velocity $\mathbf{v} = 10^{-3} \mathbf{V}$;
  change of pressure $p = 10^3 P$;
  the dynamic viscosity is about $4 \times 10^{-3}\ Pa\ sec$;
  the density is $10^3\ kg/m^3$.

The Navier-Stokes equation in new variables

$$\frac{\partial \mathbf{V}}{\partial t} - 4\Delta_X \mathbf{V} + 0.5(\mathbf{V}, \nabla_X)\mathbf{V} + \nabla_X P = 0,\ \nabla \cdot \mathbf{V} = 0.$$

Different scalings of the Navier–Stokes equations

- The characteristic length is $10^{-2}\ m$;
  the characteristic diameter is $10^{-3}\ m$ ($\varepsilon = 0.1$);
  change of space variable $X = 10^{-2}x$;
  change of velocity $\mathbf{v} = 10^{-4}\mathbf{V}$;
  change of pressure $p = 10^2 P$;
  the dynamic viscosity is about $4 \times 10^{-3}\ Pa\ sec$;
  the density is $10^3\ kg/m^3$.

The Navier-Stokes equation in new variables

$$10^2 \frac{\partial \mathbf{V}}{\partial t} - 4\Delta_X \mathbf{V} + 0.5(\mathbf{V}, \nabla_X)\mathbf{V} + \nabla_X P = 0,\ \nabla \cdot \mathbf{V} = 0.$$

Formulation of the problem

Let boundary value $\mathbf{g} \in C^{[\frac{J+1}{2}]+1}(0, 2\pi; W^{3/2,2}(\partial B_\varepsilon))$ and

$$\widetilde{F}^j(t) = \int\limits_{\gamma_\varepsilon^j} \mathbf{g} \cdot \mathbf{n} \, \mathrm{d}S \equiv \varepsilon^{n-1} F^j(t), \quad j = N_1 + 1, \ldots, N, \tag{3}$$

where $\mathbf{n}$ is the unit outward normal vector to γ_ε^j.
Compatibility condition for the flow rates $F^j(t)$:

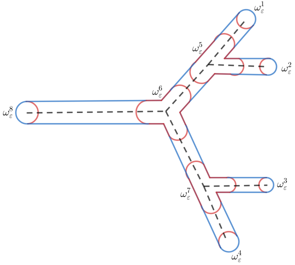
$$\sum_{j=1}^{J} F^j(t) = 0 \quad \forall t \in [0, 2\pi]. \tag{4}$$

Formulation of the problem

Let $\mathbf{g} \in C^{[\frac{J+1}{2}]+1}(0,2\pi; W^{2,2}(B_\varepsilon))$ be the divergence free time-periodic extension of the boundary function $\mathbf{g}$ satisfying for all $t \in [0,2\pi]$ the following asymptotic estimates

$$\begin{aligned} \sup_{x\in B_\varepsilon} |\mathbf{g}(x,t)| \le c, \qquad & \|\nabla \mathbf{g}\|_{L^2(B_\varepsilon)} \le c\varepsilon^{\frac{n-3}{2}} \quad \forall t\in [0,2\pi], \\ \|\mathbf{g}_t\|_{L^2(B_\varepsilon)} \le c\varepsilon^{\frac{n-1}{2}}, \quad & \|\nabla^2 \mathbf{g}\|_{L^2(B_\varepsilon)} \le c\varepsilon^{\frac{n-5}{2}} \quad \forall t\in [0,2\pi]. \end{aligned} \tag{5}$$

where the constant c is independent of ε.



Formulation of the problem

We consider the following variations problem: to find a vector-field $\mathbf{v} = \mathbf{u} + \mathbf{g}$ with $\operatorname{div} \mathbf{u} = 0$, $\mathbf{u} \in L^\infty_{\text{per}}(0, 2\pi; \mathring{W}^{1,2}(B_\varepsilon) \cap W^{2,2}(B_\varepsilon))$, $\mathbf{u}_t \in L^2_{\text{per}}(0, 2\pi; L^2(B_\varepsilon))$ satisfying the integral identity

$$\begin{aligned}\int\limits_{B_\varepsilon} \Big(\frac{1}{\varepsilon^\beta} \mathbf{u}_t \cdot \boldsymbol{\eta} + \nu \nabla \mathbf{u} : \nabla \boldsymbol{\eta} - ((\mathbf{u} + \mathbf{g}) \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{u} - (\mathbf{u} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{g} \Big) \mathrm{d}x \\ = \int\limits_{B_\varepsilon} \mathbf{f} \cdot \boldsymbol{\eta} \, \mathrm{d}x,\end{aligned} \tag{6}$$

for every divergence free vector field $\boldsymbol{\eta} \in \mathring{W}^{1,2}(B_\varepsilon)$. Here $\mathbf{g}$ is an arbitrary extension satisfying (5) and $\mathbf{f}$ is an arbitrary function such that $\mathbf{f} \in L^2(0, 2\pi; L^2(B_\varepsilon))$.
Denote

$$A_1(t) = \|\mathbf{f}(\,\cdot\,, t)\|^2_{L^2(B_\varepsilon)}. \tag{7}$$

Solvability of the problem

Theorem

Let $B_\varepsilon \subset \mathbb{R}^2,\ \partial B_\varepsilon \in C^2$. Suppose that the extended function $\mathbf{g} \in C^{[\frac{J+1}{2}]+1}(0,2\pi;W^{2,2}(B_\varepsilon))$ satisfies the conditions (3), (4), (5), and $\mathbf{f} \in L^2(0,2\pi;L^2(B_\varepsilon))$. Then for sufficiently small ε, the variational problem (6) admits a solution $\mathbf{u}$ satisfying the estimates

$$\sup_{t\in[0,2\pi]} \|\mathbf{u}(\,\cdot\,,t)\|^2_{L^2(B_\varepsilon)} + \varepsilon^\beta \int_0^{2\pi}\int_{B_\varepsilon} |\nabla \mathbf{u}(x,t)|^2 \mathrm{d}x\,\mathrm{d}t \le c\varepsilon^{2+\beta}\int_0^{2\pi} A_1(t)\,\mathrm{d}t,$$

$$c \sup_{t\in[0,2\pi]} \|\nabla\mathbf{u}(\,\cdot\,,t)\|^2_{L^2(B_\varepsilon)} + \int_0^{2\pi}\int_{B_\varepsilon} |\mathbf{u}_t(x,t)|^2\ \mathrm{d}x\,\mathrm{d}t$$

$$+\varepsilon^\beta \int_0^{2\pi}\int_{B_\varepsilon} |\nabla^2\mathbf{u}(x,t)|^2\ \mathrm{d}x\,\mathrm{d}t \le c\varepsilon^\beta \int_0^{2\pi} A_1(t)\,\mathrm{d}t$$

with constants independent of ε.

Asymptotic expansion (proposed by G. Panasenko and K. Pileckas, Nonlinear Analysis TMA 2015)

Main steps:

- First, we solve the time-periodic problem on the graph and find the macroscopic pressure.
- At the nodes, it satisfies the Kirchhoff-type junction conditions.
- We multiply the Poiseuille type velocity and pressure in every cylinder $\Pi_\varepsilon^{(e)}$ by cut-off function ζ equal to one in the middle part of the cylinder and vanishing in some $O(\varepsilon)$ - neighbourhood of the nodes.
- We construct boundary layer correctors, which compensate the residuals which we get in the previous step.

Asymptotic expansion

The asymptotic expansion of the velocity is constructed in the form

$$\mathbf{v}^{(J)}(x,t) = \sum_{i=1}^{M} \zeta\Big(\frac{x_n^{(e_i)}}{3r\varepsilon}\Big)\zeta\Big(\frac{|e_i| - x_n^{(e_i)}}{3r\varepsilon}\Big) \sum_{j=0}^{J} \varepsilon^j \mathbf{V}_j^{(e_i)}(y^{(e_i)\prime}, t)$$

$$+ \sum_{l=1}^{N} \Big(1 - \zeta\Big(\frac{|x - O_l|}{|e|_{\min}}\Big)\Big) \sum_{j=-1}^{J} \varepsilon^j \mathbf{V}_j^{[BLO_l]}(y,t),$$

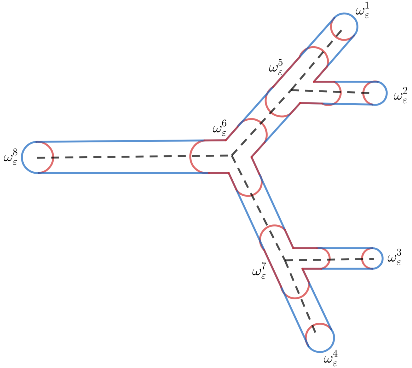
where

- $y = \dfrac{x^{(e)}}{\varepsilon}$;
- $\zeta(\tau) = \begin{cases} 0, & \tau \le \frac{1}{3} \\ 1, & \tau \ge \frac{2}{3} \end{cases}$;
- $|e|_{\min}$ is the minimal length of the edges;
- $r = 3\max\{\mathrm{diam}\ \sigma_1, \dots, \mathrm{diam}\ \sigma_{\mathrm{M}}\} + 1$;
- $\mathbf{V}_j^{(e_i)}(y^{(e_i)\prime}, t)$ - the Poiseuille type velocities;
- $\mathbf{V}_j^{[BLO_l]}(y,t)$ - the boundary layer terms.

Asymptotic expansion

$$
\mathbf{v}^{(J)}(x,t) = \sum_{i=1}^{M} \zeta\Big(\frac{x_n^{(e_i)}}{3r\varepsilon}\Big)\zeta\Big(\frac{|e_i| - x_n^{(e_i)}}{3r\varepsilon}\Big)\sum_{j=0}^{J} \varepsilon^j \mathbf{V}_j^{(e_i)}(y^{(e_i)\prime}, t)
$$

$$
+ \sum_{l=1}^{N} \Big(1 - \zeta\Big(\frac{|x - O_l|}{|e|_{\min}}\Big)\Big) \sum_{j=-1}^{J} \varepsilon^j \mathbf{V}_j^{[BLO_l]}(y,t),
$$



Asymptotic expansion

The asymptotic expansion of the pressure has the similar form:

$$p^{(J)}(x,t) = \sum_{i=1}^{M} \zeta\Big(\frac{x_n^{(e_i)}}{3r\varepsilon}\Big)\zeta\Big(\frac{|e_i| - x_n^{(e_i)}}{3r\varepsilon}\Big)\sum_{j=0}^{J} \varepsilon^{j-2}\big(- s_j^{(e_i)}(t)x_n^{(e_i)} + a_j^{(e_i)}(t)\big)$$

$$+ \sum_{l=1}^{N} \Big(1 - \zeta\Big(\frac{|x - O_l|}{|e|_{\min}}\Big)\Big) \sum_{j=-1}^{J} \varepsilon^{j-1} P_j^{[BLO_l]}(y,t).$$

Asymptotic expansion: problem on the graph

Find a function $p_0 \in L^2_{\mathrm{per}}(0, 2\pi; W^{1,2}(\mathcal{B}))$ such that equations

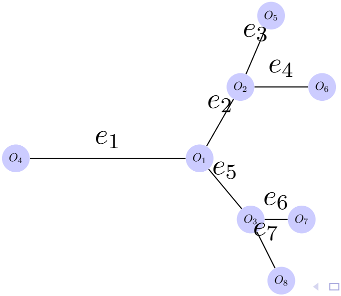
$$\begin{cases} -\dfrac{\partial}{\partial x_n^{(e)}}\Big(L^{(e)}\dfrac{\partial p_0}{\partial x_n^{(e)}}(x_n^{(e)}, t)\Big) = 0, & x_n^{(e)} \in (0, |e|), \forall e = e_j,\ j = 1, \dots, M, \\ -\displaystyle\sum_{e: O_l \in e}\Big(L^{(e)}\frac{\partial p_0}{\partial x_n^{(e)}}\Big)(0, t) = 0, & l = 1, \dots, N_1, \\ \qquad -\Big(L^{(e)}\dfrac{\partial p_0}{\partial x_n^{(e)}}\Big)(0, t) = \Psi_l(t), & l = N_1 + 1, \dots, N, \end{cases}$$

hold. Here $\Psi_l(t) = \displaystyle\int_{\gamma^l} \mathbf{g}_l \cdot \mathbf{n}\, \mathrm{d}S$, $p_0^{(e)}(x_n^{(e)}, t) = -s_0^{(e)}(t) x_n^{(e)} + a_0^{(e)}(t)$.

Operator $L^{(e)}$ relates the pressure slope $\mathcal{S}$ and the flux $\mathcal{H}$ in an infinite cylindrical pipe with section $\sigma^{(e)}$.

Asymptotic expansion: problem on the graph

$$\begin{cases} -\dfrac{\partial}{\partial x_n^{(e)}}\Big(L^{(e)}\dfrac{\partial p_0}{\partial x_n^{(e)}}(x_n^{(e)},t)\Big)=0, & x_n^{(e)}\in(0,|e|), \forall e=e_j,\ j=1,\dots,M, \\ -\displaystyle\sum_{e:O_l\in e}\Big(L^{(e)}\frac{\partial p_0}{\partial x_n^{(e)}}\Big)(0,t)=0, & l=1,\dots,N_1, \\ \qquad -\Big(L^{(e)}\dfrac{\partial p_0}{\partial x_n^{(e)}}\Big)(0,t)=\Psi_l(t), & l=N_1+1,\dots,N, \end{cases}$$



Asymptotic expansion: operator $L^{(e)}$

In order to find the operator $L^{(e)}$, we consider the following periodic in time boundary value problem for the heat equation: for given $\mathcal{S} \in L^2_{\text{per}}(0, 2\pi)$ find $\mathcal{V} \in L^2_{\text{per}}(0, 2\pi; \mathring{W}^{1,2}(\sigma^{(e)}))$ with $\dfrac{\partial \mathcal{V}}{\partial t} \in L^2_{\text{per}}(0, 2\pi; L^2(\sigma^{(e)}))$ such that

$$\begin{cases} \dfrac{\partial \mathcal{V}}{\partial t}(y^{(e)'}, t) - \nu \Delta'_{y^{(e)'}} \mathcal{V}(y^{(e)'}, t) = S(t), \quad y^{(e)'} \in \sigma^{(e)}, \ t > 0, \\ \mathcal{V}(y^{(e)'}, t)|_{\partial \sigma^{(e)}} = 0, \quad \mathcal{V}(y^{(e)'}, t) = \mathcal{V}(y^{(e)'}, t + 2\pi) \end{cases}$$

and denote

$$L^{(e)} \mathcal{S}(t) = \int\limits_{\sigma^{(e)}} \mathcal{V}(y^{(e)'}, t)\, \mathrm{d}y^{(e)'} = \mathcal{H}(t).$$

$L^{(e)}$ is bounded linear operator acting from $L^2_{\text{per}}(0, 2\pi)$ to $W^{1,2}_{\text{per}}(0, 2\pi)$.

Asymptotic expansion: boundary layer terms

The boundary layer terms $(\mathbf{V}_0^{[BLO_l]}, P_0^{[BLO_l]})$ are defined as a solution of the periodic in time Stokes problem in the unbounded domain Ω_l:

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{V}_0^{[BLO_l]} - \nu\Delta_y \mathbf{V}_0^{[BLO_l]} + \nabla_y P_0^{[BLO_l]} \\ = \displaystyle\sum_{e:O_l\in e} \Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\frac{\partial}{\partial t}V_0^{(e)}(y^{(e)\prime},t) + \nu\frac{\partial^2}{\partial y_n^{(e)2}}\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big)\mathbf{V}_0^{(e)}(y^{(e)\prime},t) \\ + \nabla_y\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)y_n^{(e)}\Big)s_0^{(e)}(t) - \nabla_y\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big)\hat{a}_1^{(e)}(t)\Big), \quad y\in\Omega_l, \\ \mathrm{div}_y\,\mathbf{V}_0^{[BLO_l]} = -\displaystyle\sum_{e:O_l\in e}\frac{\partial}{\partial y_n^{(e)}}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)V_{0,n}^{(e)}(y^{(e)\prime},t), \quad y\in\Omega_l, \\ \mathbf{V}_0^{[BLO_l]}|_{\partial\Omega_l} = 0, \quad \mathbf{V}_0^{[BLO_l]}(y,t) = \mathbf{V}_0^{[BLO_l]}(y,t+2\pi). \end{cases}$$

Justification of the asymptotic

Represent $\mathbf{v}$, p as the sums $\mathbf{v} = \mathbf{u} + \mathbf{u}^{(J)} = \mathbf{u} + \mathbf{v}^{(J)} + \mathbf{w}^{(J)}$, $p = q + p^{(J)}$, where $\mathbf{w}^{(J)} \in L^2_{\mathrm{per}}(0, 2\pi; W^{2,2}(B_\varepsilon) \cap \mathring{W}^{1,2}(B_\varepsilon))$ and $\operatorname{div} \mathbf{w}^{(J)} = -h^{(j)} = -\operatorname{div} \mathbf{v}^{(J)}$. Then $\mathbf{u}^{(J)} \in L^2_{\mathrm{per}}(0, 2\pi; W^{2,2}(B_\varepsilon))$, $\mathbf{u}_t^{(J)} \in L^2_{\mathrm{per}}(0, 2\pi; L^2(B_\varepsilon))$. The difference $\mathbf{u} = \mathbf{v} - \mathbf{u}^{(J)}$ is divergence free, satisfies the periodicity condition, the boundary condition $\mathbf{u}(x,t)|_{\partial B_\varepsilon} = 0$ and the integral identity

$$\int\limits_{B_\varepsilon} \Big(\frac{1}{\varepsilon^\beta} \mathbf{u}_t \cdot \boldsymbol{\eta} + \nu \nabla \mathbf{u} : \nabla \boldsymbol{\eta} - ((\mathbf{u} + \mathbf{u}^{(J)}) \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{u} - (\mathbf{u} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{u}^{(J)} \Big) \, \mathrm{d}x$$

$$= \int\limits_{B_\varepsilon} \mathbf{f}_1^{(J)} \cdot \boldsymbol{\eta} \, \mathrm{d}x$$

for every $\boldsymbol{\eta} \in H(B_\varepsilon)$.

Theorem

Let $n = 3$. The following estimates

$$\sup_{t\in[0,\,2\pi]} \|\mathbf{u}(\,\cdot\,,t)\|^2_{L^2(B_\varepsilon)} + \varepsilon^\beta \int_0^{2\pi} \int_{B_\varepsilon} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \le c\varepsilon^{2J-2+\beta},$$

$$\sup_{t\in[0,2\pi]} \|\nabla\mathbf{u}(\,\cdot\,,t)\|^2_{L^2(B_\varepsilon)} + \int_0^{2\pi} \int_{B_\varepsilon} |\mathbf{u}_t|^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^\beta \int_0^{2\pi} \int_{B_\varepsilon} |\nabla^2 \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\le c\varepsilon^{2J-4+\beta}$$

hold.

Justification of the asymptotic

Theorem

Moreover, there exists the pressure function $q \in L^2_{\mathrm{per}}(0, 2\pi; L^2(B_\varepsilon))$ such that $\int\limits_{B_\varepsilon} q(x,t)\,\mathrm{d}x = 0$ and

$$\int\limits_{B_\varepsilon}\Big(\frac{1}{\varepsilon^\beta}\mathbf{u}_t\cdot\boldsymbol{\eta} + \nu\nabla\mathbf{u} : \nabla\boldsymbol{\eta} - ((\mathbf{u}+\mathbf{u}^{(J)})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u} - (\mathbf{u}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}^{(J)}\Big)\,\mathrm{d}x$$

$$= \int\limits_{B_\varepsilon} q\,\mathrm{div}\,\boldsymbol{\eta}\,\mathrm{d}x + \int\limits_{B_\varepsilon}\mathbf{f}_1^{(J)}\cdot\boldsymbol{\eta}\,\mathrm{d}x, \quad \forall\boldsymbol{\eta} \in \mathring{W}^{1,2}(B_\varepsilon).$$

If $J \geq 2$, then the following estimate

$$\int\limits_0^{2\pi}\int\limits_{B_\varepsilon} |q|^2\,\mathrm{d}x\,\mathrm{d}t \leq c\varepsilon^{2J-4-\beta}$$

holds.

Steady-state Navier–Stokes equations with Bernoulli pressure

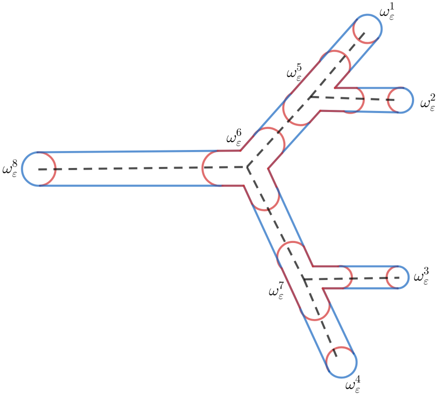
The steady-state Navier–Stokes equations in a tube structure B_ε

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0, & x \in B_\varepsilon, \\ \operatorname{div}\mathbf{v} = 0, & x \in B_\varepsilon, \\ \mathbf{v} = 0, & x \in \partial B_\varepsilon \setminus \cup_{j=N_1+1}^{N} \gamma_\varepsilon^j, \\ \mathbf{v}_{\boldsymbol{\tau}} = 0, & x \in \gamma_\varepsilon^j, \\ -\nu\partial_n(\mathbf{v}\cdot\mathbf{n}) + \left(p + \frac{1}{2}|\mathbf{v}|^2\right) = c_j/\varepsilon^2, & x \in \gamma_\varepsilon^j,\ j = N_1+1,\dots,N, \end{cases}$$

where ν is a positive constant, $\mathbf{n}$ is the unit normal vector to γ_ε^j, $\mathbf{v}_{\boldsymbol{\tau}} = \mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}$ is the tangential component of the vector $\mathbf{v}$, $\partial_n\mathbf{v} = \nabla\mathbf{v}\cdot\mathbf{n}$ is the normal derivative of $\mathbf{v}$, c_j are some constants.

Definition of a tube structure

Suppose that it is a connected set and that the boundary ∂B_ε of B_ε is C^2-smooth everywhere except for the parts of the boundary which are the bases $\gamma_\varepsilon^j = \{x^{(e)'} \in \sigma^{O_j}, x_n^{(e)} = 0\}$ of cylinders $\Pi_\varepsilon^{(e)}$.



Steady-state Navier–Stokes equations with Bernoulli pressure

We can rewrite steady-state Navier-Stokes equation with the right-hand side $\mathbf{f} \in L^2(B_\varepsilon)$ in the following form

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mathbf{v}\cdot(\nabla\mathbf{v})^t + \nabla\Phi = \mathbf{f}, \quad x \in B_\varepsilon, \\ \operatorname{div}\mathbf{v} = 0, \ x \in B_\varepsilon, \\ \mathbf{v} = 0, \ x \in \partial B_\varepsilon \setminus \cup_{j=N_1+1}^{N} \gamma_\varepsilon^j, \\ \mathbf{v}_{\boldsymbol{\tau}} = 0, \ x \in \gamma_\varepsilon^j, \\ \Phi = p_j, x \in \gamma_\varepsilon^j, \ j = \overline{N_1+1, N}, \end{cases} \tag{8}$$

where $\Phi = (p + \frac{1}{2}|\mathbf{v}|^2)$ is the Bernoulli pressure, p_j stand for the constants c_j/ε^2.

Steady-state Navier-Stokes equations with Bernoulli pressure

Let $\Gamma = \partial B_\varepsilon \setminus \cup_{j=N_1+1}^{N} \gamma_\varepsilon^j$ be the lateral surface of the domain B_ε, then

$\widehat{W}_\gamma^{1,2}(B_\varepsilon) = \{\boldsymbol{\eta} \in W^{1,2}(B_\varepsilon) : \boldsymbol{\eta}|_\Gamma = 0,\ \boldsymbol{\eta}_\tau|_{\gamma_\varepsilon^j} = 0,\ j = N_1{+}1, \dots, N\}.$

Definition

A weak solution of (8) problem is a vector field $\mathbf{v} \in \widehat{J}_\gamma^{1,2}(B_\varepsilon) = \{\boldsymbol{\eta} \in \widehat{W}_\gamma^{1,2}(B_\varepsilon) : \operatorname{div} \boldsymbol{\eta} = 0\}$, satisfying the integral identity

$$\nu \int\limits_{B_\varepsilon} \nabla \mathbf{v} : \nabla \boldsymbol{\eta} \, \mathrm{d}x + \int\limits_{B_\varepsilon} (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \boldsymbol{\eta} \, \mathrm{d}x - \int\limits_{B_\varepsilon} (\boldsymbol{\eta} \cdot \nabla)\mathbf{v} \cdot \mathbf{v} \, \mathrm{d}x$$

$$= - \sum_{j=N_1+1}^{N} p_j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n} \, \mathrm{d}x' + \int\limits_{B_\varepsilon} \mathbf{f} \cdot \boldsymbol{\eta} \, \mathrm{d}x$$

for every $\boldsymbol{\eta} \in \widehat{J}_\gamma^{1,2}(B_\varepsilon)$.

Steady-state Navier-Stokes equations

Introduce $p_j^* = p_j - p_N, \ j = N_1, \dots, N$. We get an equivalent weak formulation: find a vector field $\mathbf{v} \in \widehat{J}_\gamma^{1,2}(B_\varepsilon)$ satisfying the integral identity

$$\begin{aligned} \nu \int\limits_{B_\varepsilon} \nabla \mathbf{v} : \nabla \boldsymbol{\eta} \, \mathrm{d}x + \int\limits_{B_\varepsilon} (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \boldsymbol{\eta} \, \mathrm{d}x - \int\limits_{B_\varepsilon} (\boldsymbol{\eta} \cdot \nabla)\mathbf{v} \cdot \mathbf{v} \, \mathrm{d}x \\ = - \sum_{j=N_1+1}^{N-1} p_j^* \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n} \, \mathrm{d}x' + \int\limits_{B_\varepsilon} \mathbf{f} \cdot \boldsymbol{\eta} \, \mathrm{d}x \end{aligned} \tag{9}$$

for every $\boldsymbol{\eta} \in \widehat{J}_\gamma^{1,2}(B_\varepsilon)$.

Existence of the weak solution

Theorem

For arbitrary $\mathbf{f} \in L^2(B_\varepsilon)$ and $p_j^* \in \mathbb{R}$, $j = N_1+1, \ldots, N-1$ problem (8) admits at least one weak solution $\mathbf{v} \in \widehat{J}_\gamma^{1,2}(B_\varepsilon)$. There holds the estimate

$$\|\nabla \mathbf{v}\|_{L^2(B_\varepsilon)} \leq c\Big(\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p_j^*| + \varepsilon \|\mathbf{f}\|_{L^2(B_\varepsilon)}\Big)$$

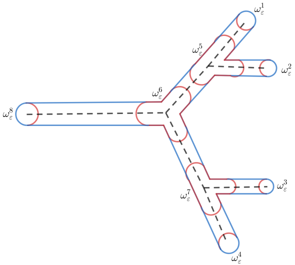
with the constant c independent of ε.

Asymptotic expansion of the solution

$$\mathbf{v}^{(J)}(x) = \sum_{O_l,\ l=N_1+1,\dots,N;\ e=\overline{O_l O_{i_l}}} \zeta\Big(\frac{x_n^{(e)}}{3r\varepsilon}\Big)\mathbf{V}^{[e,J]}\Big(\frac{x^{(e)'}}{\varepsilon}\Big)$$

$$+ \sum_{e=\overline{O_\alpha O_\beta};\ \alpha,\beta\le N_1} \zeta\Big(\frac{x_n^{(e)}}{3r\varepsilon}\Big)\zeta\Big(\frac{|e|-x_n^{(e)}}{3r\varepsilon}\Big)\mathbf{V}^{[e,J]}\Big(\frac{x^{(e)'}}{\varepsilon}\Big)$$

$$+ \sum_{l=1}^{N}\Big(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\Big)\mathbf{V}^{[BLO_l,J]}\Big(\frac{x-O_l}{\varepsilon}\Big),$$



Asymptotic expansion of the solution

The asymptotic expansion of the pressure for every half-cylinder $\Pi_\varepsilon^{(e)}$, $x_n < |e|/2$, corresponding to the edge $e = \overline{O_l O_{i_l}}$, $l = N_1 + 1, \ldots, N$, (O_l is the origin of the local coordinate system) is sought in the form:

$$p^{(J)}(x) = -s^{(e)} x_n^{(e)} + a^{(e)} + \frac{1}{\varepsilon}\left(1 - \zeta\Big(\frac{|x - O_l|}{|e|_{\min}}\Big)\right) P^{[BLO_l,J]}\Big(\frac{x - O_l}{\varepsilon}\Big),$$

and on every half-bundle HB_{O_l}, $l = 1, \ldots, N_1$, (O_l is the origin of the local coordinate system) we define:

$$p^{(J)}(x) = \sum_{e \subset \mathcal{B}_l} \zeta\Big(\frac{x_n^{(e)}}{3r\varepsilon}\Big)\big(- s^{(e)} x_n^{(e)} + a^{(e)} - a^{(e_s)}\big) + a^{(e_s)}$$

$$+\frac{1}{\varepsilon}\left(1 - \zeta\Big(\frac{|x - O_l|}{|e|_{\min}}\Big)\right) P^{[BLO_l,J]}\Big(\frac{x - O_l}{\varepsilon}\Big).$$

Solve the conductivity problem on the graph for the function p_0:

$$\begin{cases} -\kappa_e \dfrac{\partial^2 p_0^{(e)}}{\partial {x_n^{(e)}}^2}(x_n^{(e)}) = 0, & x_n^{(e)} \in (0, |e|), \\ -\displaystyle\sum_{e: O_l \in e} \kappa_e \dfrac{\partial p_0^{(e)}}{\partial x_n^{(e)}}(0) = 0, & l = 1, \ldots, N_1, \\ p_0^{(e)}(0) = c_l, & l = N_1 + 1, \ldots, N, \\ p_0^{(e)}(0) = p_0^{(e_s)}(0), & \forall e \subset \mathcal{B}_l. \end{cases}$$

Asymptotic expansion of the solution (the base case)

Solving the above conductivity problem, we define for every edge e the constants $s_0^{(e)}$ and $a_0^{(e)}$ such that

$$p_0^{(e)}(x^{(e)}) = -s_0^{(e)} x_n^{(e)} + a_0^{(e)}$$

and the velocity $V_0^{(e)}(y^{(e)\prime})$ is the solution of the Dirichlet problem

$$\begin{cases} -\nu\Delta_{(y^{(e)\prime})} V_0^{(e)}(y^{(e)\prime}) = 1, & y^{(e)\prime} \in \sigma^{(e)}; \\ V_0^{(e)}(y^{(e)\prime}) = 0, & y^{(e)\prime} \in \partial\sigma^{(e)}, \end{cases}$$

and

$$\kappa_e = \int\limits_{\sigma^{(e)}} V_0^{(e)}(y^{(e)\prime}) \mathrm{d}y^{(e)\prime}.$$

Asymptotic expansion of the solution (the base case)

For $l = 1, \dots, N_1$ the boundary layer problem for $(\mathbf{V}_0^{[BLO_l]}(y), P_0^{[BLO_l]}(y))$ is:

$$\begin{cases} -\nu \Delta_y \mathbf{V}_0^{[BLO_l]} + \nabla_y P_0^{[BLO_l]} = \mathbf{f}_0^{[REGO_l]} + \mathbf{f}_0^{[BLO_l]}, & y \in \Omega_l, \\ \mathrm{div}_y \mathbf{V}_0^{[BLO_l]} = h_0^{[REGO_l]}, & y \in \Omega_l, \\ \mathbf{V}_0^{[BLO_l]} = 0, & y \in \partial\Omega_l. \end{cases}$$

Error estimate

Set $\widetilde{\mathbf{v}}^{(J)} = \mathbf{v}^{(J)} + \mathbf{w}^{(J)}$, where $\mathbf{w}^{(J)} \in \mathring{W}^{1,2}(B_\varepsilon)$ is a vector field such that $\operatorname{div} \mathbf{w}^{(J)} = -h^{(j)}$.

Theorem

The following error estimate

$$\|\mathbf{v} - \widetilde{\mathbf{v}}^{(J)}\|_{W^{1,2}(B_\varepsilon)} = O(\varepsilon^{J+(n-1)/2})$$

holds.

Conclusion

- The results obtained for the time-periodic Stokes system was generalized by K. Kaulakytė and K. Pileckas.
- Constructed asymptotic expansion for the Navier-Stokes equations let create hybrid dimension models. These models reduce the numerical simulation cost and may be used to create a simplified blood circulation model for small and very small vessels.
- Obtained results may be developed for more complicated cases, time-periodic case with given Bernoulli pressure etc.

Publications

1. R. Juodagalvytė, K. Kaulakytė, Time periodic boundary value Stokes problem in a domain with an outlet to infinity, Nonlinear Analysis: Modelling and Control 23(6), 866–888 (2018).
2. R. Juodagalvytė, G. Panasenko, K. Pileckas, Time periodic Navier–Stokes equations in a thin tube structure, Boundary Value Problems 2020(1), 1–35 (2020).
3. R. Juodagalvytė, G. Panasenko, K. Pileckas, Steady-state Navier–Stokes equations in thin tube structure with the Bernoulli pressure inflow boundary conditions: Asymptotic analysis, Mathematics 9(19), 1–20 (2021).

Dissemination

The results of this thesis were presented at the following seminar and conferences

1. Seminar "Journées Scientifiques GDF MORPHEA, École des Mines de Saint-Etienne", June 6, 2019, Saint-Etienne, France.
2. International conference "Journée EDP Auverge-Rhône-Alpes", November 7-8, 2019, Chambery, France.
3. International conference "8th European Congress of Mathematic" minisymposium "Multiscale Modeling and Methods: Application in Engineering, Biology and Medicine", June 20-26, 2020, Portorož, Slovenia.