Discrete Sturm–Liouville problem with Two-Point Nonlocal Boundary Condition and Natural Approximation of a Derivative in Boundary Condition

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$$-u'' = \lambda u, \quad t \in (0,1), \tag{1}$$

with Dirichlet BC or the natural BC:

(Case *d*) $$u(0) = 0, \tag{2a}$$
(Case *n*) $$u'(0) = 0, \tag{2b}$$

and two–point NBC: [1]

(Case 1) $$u(1) = \gamma u(\xi), \tag{3a}$$
(Case 2) $$u'(1) = \gamma u'(\xi), \tag{3b}$$
(Case 3) $$u(1) = \gamma u'(\xi), \tag{3c}$$
(Case 4) $$u'(1) = \gamma u(\xi), \tag{3d}$$

[1] S. Pečiulytė, PhD thesis

The general solution of this equation $-u'' = \lambda u, \quad t \in (0,1)$ is

$$u(t) = C_1 \cos(\pi q t) + C_2 \frac{\sin(\pi q t)}{\pi q}, \quad \lambda = \lambda(q) = (\pi q)^2. \qquad (4)$$

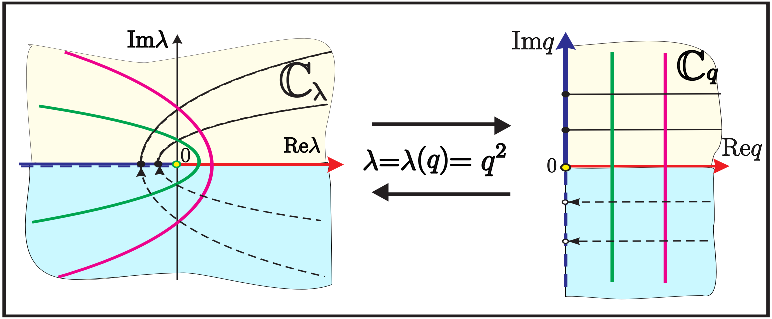


Figure: Bijective map: $\lambda = (\pi q)^2$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_q$; ∘–BP, ∘–RP.

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, n-1}, U_0 = 0, \tag{5}$$

$$U_0 = 0, \qquad \big(u(0) = 0\big) \tag{6a}$$

$$U_0 = U_1, \qquad \big(u'(0) = 0\big). \tag{6b}$$

$$U_n = \gamma U_m, \qquad \big(u(1) = \gamma u(\xi)\big) \tag{7a}$$

$$\frac{U_n - U_{n-1}}{h} = \gamma \frac{U_{m+1} - U_{m-1}}{2h}, \qquad \big(u'(1) = \gamma u'(\xi)\big), \tag{7b}$$

$$U_n = \gamma \frac{U_{m+1} - U_{m-1}}{2h}, \qquad \big(u(1) = \gamma u'(\xi)\big), \tag{7c}$$

$$\frac{U_n - U_{n-1}}{h} = \gamma U_m, \qquad \big(u'(1) = \gamma u(\xi)\big). \tag{7d}$$

and $h = 1/n$, $\xi = mh = m/n$. The truncation error is $\mathcal{O}(h)$.

For dSLP (5)–(7a,c) we have meromorphic functions

- $$U_0 = 0 \quad \text{and } U_n = \gamma U_m \quad \text{or} \quad U_n = \gamma \frac{U_{m+1} - U_{m-1}}{2h}$$

$$\gamma_c(q) := \frac{\sin(\pi q)}{\sin(\pi q \xi)}, \quad 0 < m < n \tag{8a}$$

$$\gamma_c(q) := \frac{\sin(\pi q)}{\cos(\pi q \xi)} \cdot \frac{h}{\sin(\pi q h)}, \quad 0 \le m < n. \tag{8b}$$

- $$U_0 = U_1 \quad \text{and } U_n = \gamma U_m \quad \text{or} \quad U_n = \gamma \frac{U_{m+1} - U_{m-1}}{2h}$$

$$\gamma_c(q) := \frac{\cos(\pi q(1 - h/2))}{\cos(\pi q(\xi - h/2))}, \quad 0 < m < n \tag{9a}$$

$$\gamma_c(q) := -\frac{\cos(\pi q(1 - h/2))}{\sin(\pi q(\xi - h/2))} \cdot \frac{h}{\sin(\pi q h)}, \quad 0 \le m < n, \tag{9b}$$

Discrete Sturm-Liouville Problems with $u(0) = 0$ and $u'(0) = 0$.

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = (\delta^2 U)_j + \lambda U_j = 0, \qquad (10)$$

$$(\delta^2 U)_j := \frac{(\delta U)_{j+1/2} - (\delta U)_{j-1/2}}{h_{j+1/2}} \qquad (\delta U)_{j+1/2} := \frac{U_{j+1} - U_j}{h}$$

$$(\delta U)_{-1/2} = 0, \qquad \big(u'(0) = 0\big) \qquad (11a)$$

$$(\delta U)_{n+1/2} = 0, \qquad \big(u'(1) = 0\big). \qquad (11b)$$

$j = \overline{1, n-1}$ and $h = 1/n$, $\xi = mh = m/n$. The conditions $u'(0) = 0$ and $u'(1) = 0$ truncation error is $\mathcal{O}(h^2)$. Operator $(\delta^2 U)$ can be extended to point t_0 and t_n

$$(\delta^2 U)_0 := \frac{(\delta U)_{1/2} - (\delta U)_{-1/2}}{h/2} = \frac{(\delta U)_{1/2}}{h/2} = -\lambda U_0 \qquad (12a)$$

$$(\delta^2 U)_n := \frac{(\delta U)_{n+1/2} - (\delta U)_{n-1/2}}{h/2} = \frac{(\delta U)_{n-1/2}}{h/2} = -\lambda U_n \qquad (12b)$$

$$-\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = \lambda U_j \text{ or} \tag{13}$$

$$U_{j+1} - 2zU_j + U_{j-1} = 0, \quad z = 1 - \lambda h^2/2 \tag{14}$$

and the general solution of this discrete equation have expression [2]

$$U_j = C_1 T_j(z) + C_2 \widetilde{T}_{j-1}(z), \quad j \in \mathbb{Z} \tag{15}$$

where

$$T_j(z) = \frac{(z + \sqrt{z^2 - 1})^j + (z - \sqrt{z^2 - 1})^j}{2}, \quad j \in \mathbb{Z},$$

are the Chebyshev polynomial of the first kind of degree j in z,

$$\widetilde{T}_j(z) = \frac{(z + \sqrt{z^2 - 1})^{j+1} - (z - \sqrt{z^2 - 1})^{j+1}}{2\sqrt{z^2 - 1}}, \quad j \in \mathbb{Z},$$

are the Chebyshev polynomial of the second kind of degree j in z.

[2] 1989 A.A. Samarskii and E.S. Nikolaev ”Numerical Methods for Grid Equations”

$$U_{j+1} - (\omega + \omega^{-1}) U_j + U_{j-1} = 0, \text{ where } z = z(\omega) := \frac{\omega + \omega^{-1}}{2} \quad (16)$$

and the general solution of this discrete equation is

$$U_j = C_1 W_j(\omega) + C_2 \widetilde{W}_j(\omega), \quad j \in \mathbb{Z}, \quad (17)$$

where

$$W_j(\omega) = \frac{\omega^j + \omega^{-j}}{2}, \quad \widetilde{W}_j(\omega) = \frac{\omega^j - \omega^{-j}}{\omega - \omega^{-1}}, \quad j \in \mathbb{Z}.$$

The conformal map

$$\omega^h \colon \mathbb{C}_q \to \mathbb{C}_{\omega^*}, \quad \omega = \omega^h(q) := e^{\imath \pi q h},$$

is bijection. Using maps λ_h and ω^h we construct the bijection between complex plane $\mathbb{C}_\lambda$ and domain $\mathbb{C}_q$:

$$\lambda = \lambda_h(q) := \frac{2}{h^2}\left(1 - \frac{e^{\imath \pi q h} + e^{-\imath \pi q h}}{2}\right) = \frac{4}{h^2} \sin^2 \frac{\pi q h}{2}. \quad (18)$$

The equation (10) can be rewritten in form

$$U_{j+1} - 2\cos(\pi q h)U_j + U_{j-1} = 0, \quad q \in \mathbb{C}_q^h, \tag{19}$$

and the general solution of this discrete equation is

$$U_j = C_1 \cos(\pi q t_j) + C_2 \frac{\sin(\pi q t_j)}{\sin(\pi q h)}, \quad \text{where } t_j = jh,\ j \in \mathbb{Z}. \tag{20}$$

Let us approximate natural condition $u'(0) = 0$ as [3]

$$(\delta U)_{1/2} = -h_{1/2}\lambda U_0 \tag{21}$$

and is natural condition for equation (19)

[3] 1989 A.A. Samarskii and E.S. Nikolaev "Numerical Methods for Grid Equations"

$$U_{j+1} - 2\cos(\pi qh)U_j + U_{j-1} = 0, \qquad \lambda = \frac{4}{h^2}\sin^2(\frac{\pi qh}{2})$$

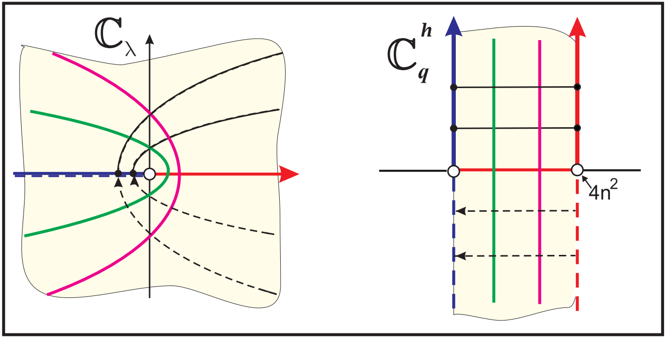


Figure: Bijective mapping $\lambda = \frac{4}{h^2}\sin^2(\frac{\pi qh}{2})$ between $\mathbb{C}_q{}^h$ and $\mathbb{C}_\lambda$.

We define grid operators:

$$\delta^+ : H(\overline{\omega}^h) \to H(\omega^h \cup \{0\}), \quad (\delta^+ U)_j := \frac{U_{j+1} - \cos(\pi q h) U_j}{h},$$

$$\delta^- : H(\overline{\omega}^h) \to H(\omega^h \cup \{n\}), \quad (\delta^- U)_j := \frac{\cos(\pi q h) U_j - U_{j-1}}{h}.$$

On the grid ω^h we have

$$(\delta^+ U)_j = (\delta^- U)_j = \big((\delta^+ U)_j + (\delta^- U)_j\big)/2 = \frac{U_{j+1} - U_{j-1}}{2h} =: (\bar{\delta} U)_j.$$

If $(\bar{\delta} U)_0 := (\delta^+ U)_0$, $(\bar{\delta} U)_n := (\delta^- U)_n$, then we have natural approximation $(\bar{\delta} U)_j$ of derivative $u'(t_j)$ on the grid $\overline{\omega}^h$.

$$-\delta^2 U = \lambda U, \quad t \in \omega^h, \tag{22}$$

with Dirichlet BC or the natural BC:

(Case *d*) $\quad U_0 = 0,$ (23a)

(Case *n*) $\quad (\bar{\delta} U)_0 = 0,$ (23b)

and two–point NBC:

(Case 1) $\quad U_n = \gamma U_m,$ (24a)

(Case 2) $\quad (\bar{\delta} U)_n = \gamma (\bar{\delta} U)_m,$ (24b)

(Case 3) $\quad U_n = \gamma (\bar{\delta} U)_m,$ (24c)

(Case 4) $\quad (\bar{\delta} U)_n = \gamma U_m,$ (24d)

where $0 \le m < n$, $\gamma \in \mathbb{R}$, $h = 1/n$, $\xi = mh = m/n$. The general solution of discrete equation (22) is

$$U_j = C_1 \cos(\pi q t_j) + C_2 \frac{\sin(\pi q t_j)}{\sin(\pi q h)}, \quad \text{where } t_j = jh,\ j \in \mathbb{Z}. \tag{25}$$

For dSLP (22)–(24) Constant Eigenvalues are equal to $\lambda_j = \lambda^h(c_j)$, where

$$c_j = Nj, \qquad j \in \mathcal{J}_\xi := \{j : j = \overline{1, K-1}\}, \qquad \text{(d1)}$$

$$c_j = N(j - 1/2), \qquad j \in \mathcal{J}_\xi := \{j : j = \overline{1, \varkappa K}\}, \qquad \text{(d2–4,n1,n3–4)}$$

$$c_j = Nj, \qquad j \in \mathcal{J}_\xi := \{j : j = \overline{0, K}\}, \qquad \text{(n2)}$$

and

$$n_{ce} = K - 1, \quad l_j = Nj, \quad k_j = Mj, \qquad \text{(d1)}$$

$$n_{ce} = \varkappa K, \quad l_j = Nj - (N-1)/2, \quad k_j = Mj - (M-1)/2, \qquad \text{(d2,n1)}$$

$$n_{ce} = K + 1, \quad l_j = Nj, \quad k_j = Mj. \qquad \text{(n2)}$$

$$n_{ce} = \varkappa K, \quad l_j = N(j - 1/2), \quad k_j = Mj - (M-1)/2, \qquad \text{(d3)}$$

$$n_{ce} = \varkappa K, \quad l_j = Nj - (N-1)/2, \quad k_j = M(j - 1/2), \qquad \text{(d4)}$$

$$n_{ce} = \varkappa K, \quad l_j = Nj - (N-1)/2, \quad k_j = Mj, \qquad \text{(n3)}$$

$$n_{ce} = \varkappa K, \quad l_j = Nj - (N/2 - 1), \quad k_j = Mj - (M-1)/2, \qquad \text{(n4)}$$

For dSLP (22)–(24) we have meromorphic functions

$$\gamma_c(q) := \frac{Z^h(q)}{P^h_\xi(q)} = \frac{\sin(\pi q)}{\sin(\pi q\xi)}, \quad 0 < m < n, \qquad \text{(d1,n2)}$$

$$\gamma_c(q) := \frac{Z^h(q)}{P^h_\xi(q)} = \frac{\cos(\pi q)}{\cos(\pi q\xi)}, \quad 0 \le m < n. \qquad \text{(d2,n1)}$$

$$\gamma_c(q) := \frac{Z^h(q)}{P^h_\xi(q)} = \frac{\sin(\pi q)}{\cos(\pi q\xi)} \cdot \frac{h}{\sin(\pi q h)}, \quad 0 \le m < n, \qquad \text{(d3)}$$

$$\gamma_c(q) := \frac{Z^h(q)}{P^h_\xi(q)} = \frac{\cos(\pi q)}{\sin(\pi q\xi)} \cdot \frac{\sin(\pi q h)}{h}, \quad 0 < m < n, \qquad \text{(d4)}$$

$$\gamma_c(q) := \frac{Z^h(q)}{P^h_\xi(q)} = -\frac{\cos(\pi q)}{\sin(\pi q\xi)} \cdot \frac{h}{\sin(\pi q h)}, \quad 0 < m < n, \qquad \text{(n3)}$$

$$\gamma_c(q) := \frac{Z^h(q)}{P^h_\xi(q)} = -\frac{\sin(\pi q)}{\cos(\pi q\xi)} \cdot \frac{\sin(\pi q h)}{h}, \quad 0 \le m < n, \qquad \text{(n4)}$$

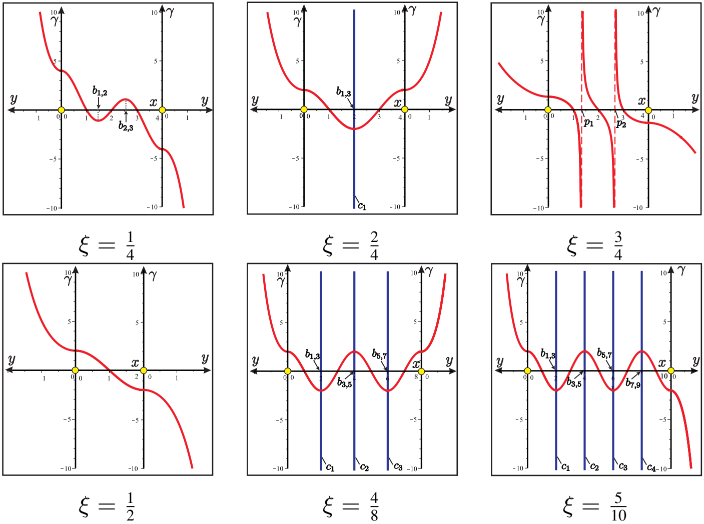


Figure: Real CF for various ξ in Case $d1, n2$.

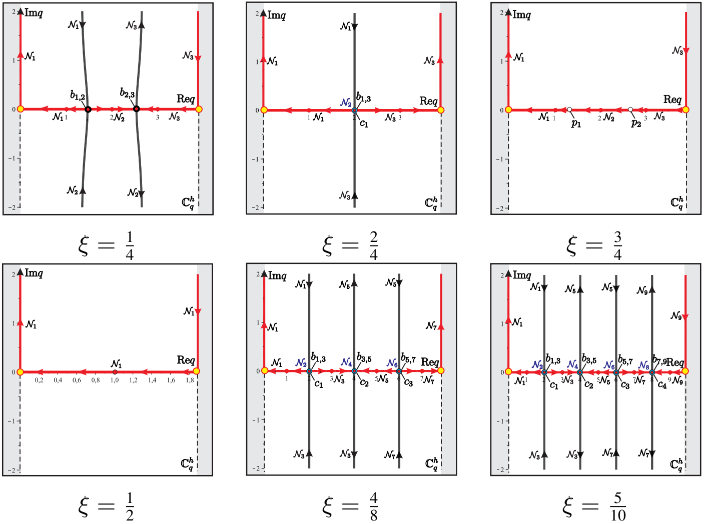
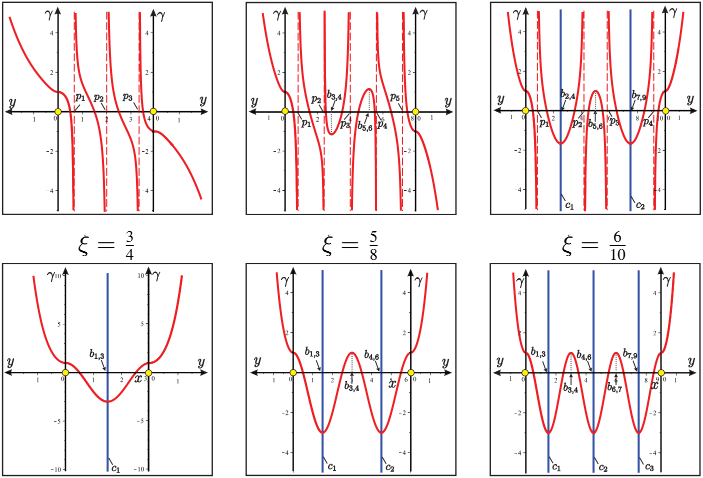


Figure: Spectrum Curves for various ξ values in Case $d1, n2$.



$\xi = \frac{3}{4}$ $\quad$ $\xi = \frac{5}{8}$ $\quad$ $\xi = \frac{6}{10}$

$\xi = \frac{1}{3}$ $\quad$ $\xi = \frac{2}{6}$ $\quad$ $\xi = \frac{3}{9}$

Figure: Real CF for various ξ in Case $d2, n1$.

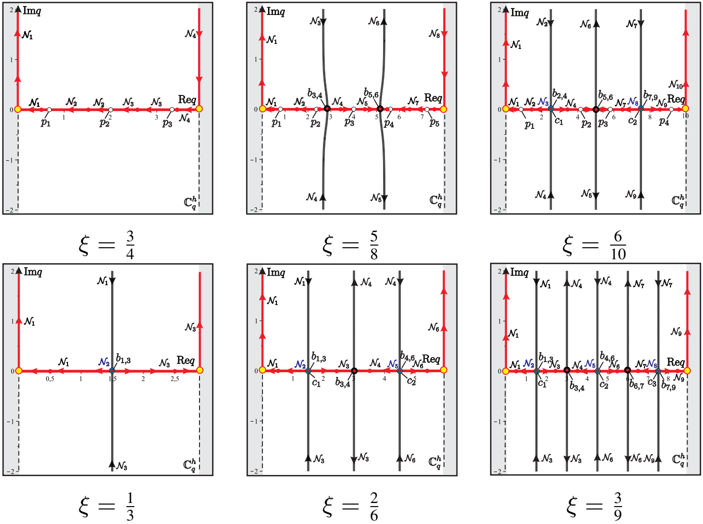
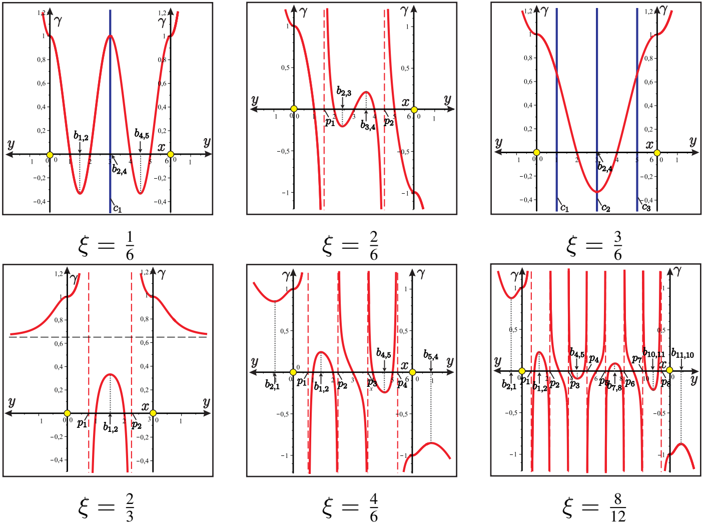


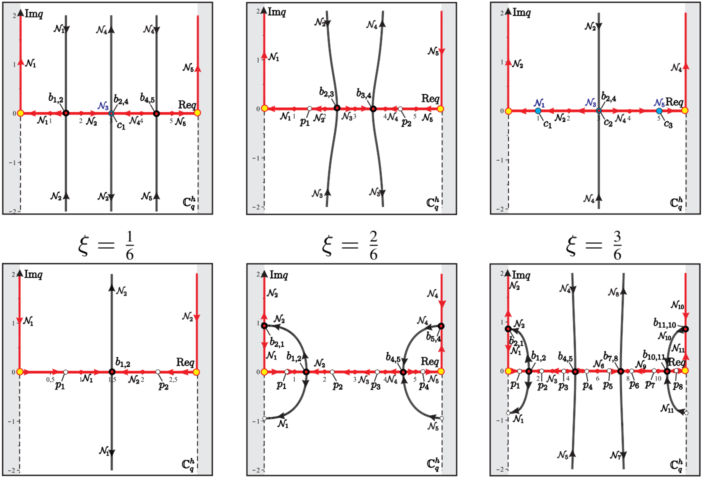
Figure: Specrum Curves for various ξ in Case $d2, n1$.



$\xi = \frac{1}{6}$ $\quad\xi = \frac{2}{6}$ $\quad\xi = \frac{3}{6}$

$\xi = \frac{2}{3}$ $\quad\xi = \frac{4}{6}$ $\quad\xi = \frac{8}{12}$

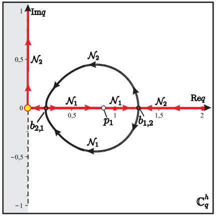
Figure: Real CF for various ξ in Case $d3$.



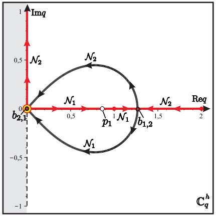
$\xi = \frac{1}{6}$ $\xi = \frac{2}{6}$ $\xi = \frac{3}{6}$

$\xi = \frac{2}{3}$ $\xi = \frac{4}{6}$ $\xi = \frac{5}{10}$

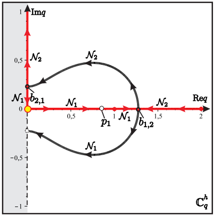
Figure: Spectrum Curves for various ξ values in Case $d3$.



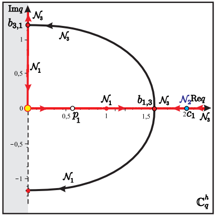
$\xi = \frac{36}{63}$



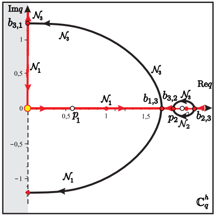
$\xi = \frac{4}{7}$



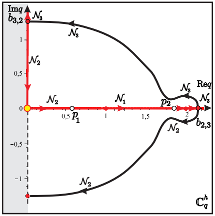
$\xi = \frac{37}{63}$



$\xi = \frac{300}{400}$

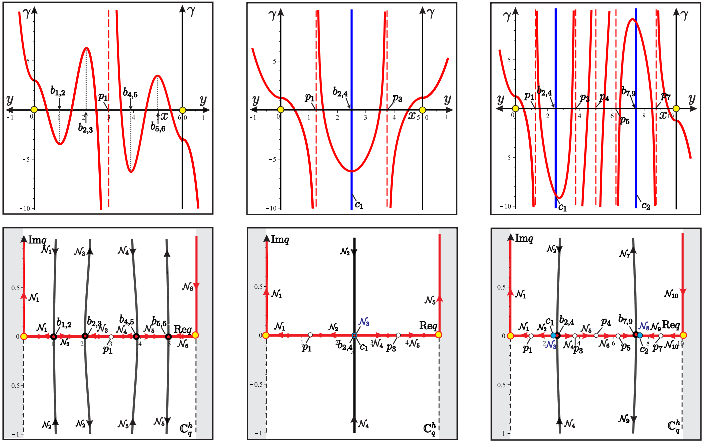


$\xi = \frac{301}{400}$



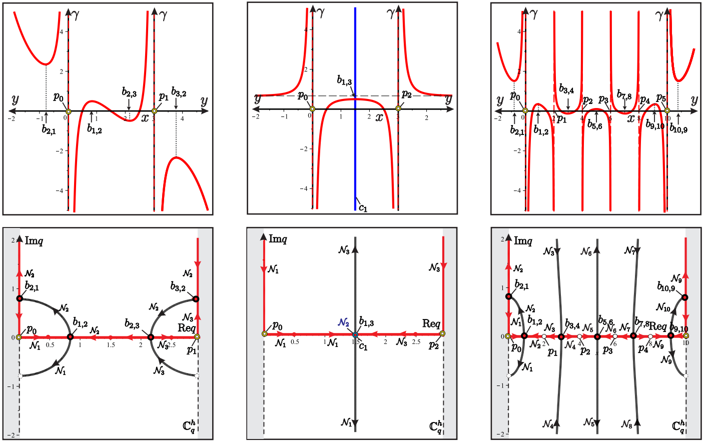
$\xi = \frac{302}{400}$

Figure: Spectrum Curves for various ξ values in Case $d3$.



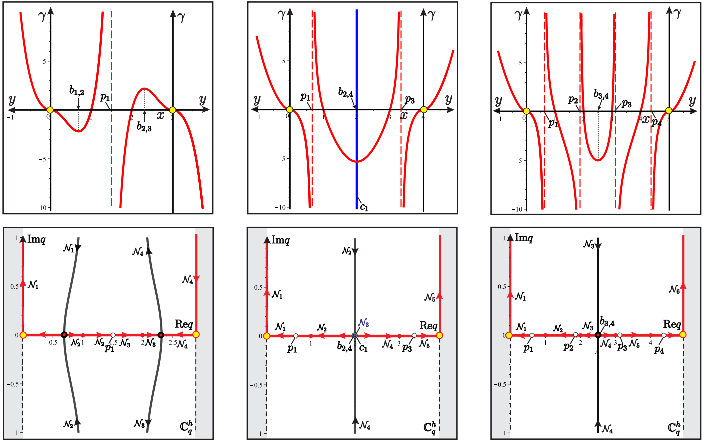
$\xi = \frac{2}{6}$ $\xi = \frac{4}{5}$ $\xi = \frac{8}{10}$

Figure: Spectrum Curves for various ξ values in Case $d4$.



$\xi = \frac{1}{3}$ $\xi = \frac{2}{3}$ $\xi = \frac{5}{10}$

Figure: Spectrum Curves for various ξ values in Case $n3$.



$\xi = \frac{1}{3}$ $\xi = \frac{3}{4}$ $\xi = \frac{4}{5}$

Figure: Spectrum Curves for various ξ values in Case $n4$.

- In case d1–2, n1–2 if $n - m = 1$ we have only real eigenvalues points.
- In cases d1,n2 the eigenvalue $\lambda = 0$ exist only if $\gamma = \frac{1}{\xi}$ and $\lambda = 4n^2$ exist only if $\gamma = (-1)^{n-m}\frac{1}{\xi}$.
- In cases d2,n1 the eigenvalue $\lambda = 0$ exist only if $\gamma = 1$ and $\lambda = 4n^2$ exist only if $\gamma = (-1)^{n-m}$.
- In cases d3 the eigenvalue $\lambda = 0$ exist only if $\gamma = 1$ and $\lambda = 4n^2$ exist only if $\gamma = (-1)^{n-m+1}$.
- In case d4 the eigenvalue $\lambda = 0$ exist only if $\gamma = \frac{1}{\xi}$ and $\lambda = 4n^2$ exist only if $\gamma = (-1)^{n-m+1}\frac{1}{\xi}$.
- In case n3 the eigenvalues $\lambda = 0$ and $\lambda = 4n^2$ do not exist.
- In case n4 the eigenvalues $\lambda = 0$ and $\lambda = 4n^2$ exist if $\gamma = 0$.