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APPLICATIONS OF NAVIER-STOKES EQUATIONS IN HEMODYNAMICS

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THE STRUCTURE OF TALK

- Time-periodic Poiseuille-type solution with minimally regular flow rate
- Poiseuille-type approximations for axisymmetric flow in a thin tube with thin stiff elastic wall
- Efficient computation of blood velocity in the left atrial appendage: A practical perspective

Time-periodic Poiseuille-type solution with minimally regular flow rate

We consider time-periodic Navier–Stokes problem

$$\left\{\begin{array}{rcl} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &=& 0, \\ \operatorname{div}\,\mathbf{u} &=& 0, \\ \mathbf{u}|_{\partial\Pi\times(0,2\pi)} &=& 0, \\ \mathbf{u}(x,0) &=& \mathbf{u}(x,2\pi), \end{array}\right. \qquad (1)$$

in an infinite cylinder $\Pi = \{x = (x_1, x_2, x_3) : x' = (x_1, x_2) \in \sigma, x_3 \in \mathbb{R}^3\}$.

- $\mathbf{u}$ - velocity of the fluid,
- p - pressure of the fluid,
- ν - viscosity of the fluid.

We look for the solution satisfying the flux condition:

$$\int_\sigma U(x',t)\,dx' = F(t), \quad F(0) = F(2\pi). \qquad (2)$$

We look for the solution ($\mathbf{u}$(x, t), p(x, t)) of problem in the form

$$\mathbf{u}(x,t) = (0,\ldots,0,U_n(x',t)), \qquad p(x,t) = -q(t)x_n + p_0(t), \quad (3)$$

By substituting (3) into (1) we obtain the following problem on the cross-section σ:

$$\begin{aligned} &U_t(x',t) - \nu\Delta' U(x',t) = q(t), \\ &U(x',t)\big|_{\partial\sigma} = 0, \quad U(x',0) = U(x',2\pi), \end{aligned} \qquad (4)$$

where $U(x',t) = U_n(x',t)$ and $q(t)$ are unknown functions, Δ' is the Laplace operator with respect to x'.

The Poiseuille flow can be uniquely determined either prescribing the pressure drop q(t) or the flow-rate F(t). However, in the real life applications the pressure is unknown, and only the flow-rate (flux) of the fluid is given. Therefore, it is necessary to prescribe the additional condition

$$\int_\sigma U(x',t)\,dx' = F(t), \quad F(0) = F(2\pi). \qquad (5)$$

Time-periodic Navier-Stokes problem in infinite cylinder Π was reduced to the following problem on the cross-section σ:

$$U_t(x',t) - \nu\Delta' U(x',t) = q(t),$$
$$U(x',t)\big|_{\partial\sigma} = 0, \quad U(x',0) = U(x',2\pi), \qquad (6)$$
$$\int\limits_{\sigma} U(x',t)\,dx' = F(t), \quad F(0) = F(2\pi).$$

We have to solve for *U(x′, t)* and *q(t)* the *inverse* parabolic problem, i.e., for *given F(t) to find a pair of functions (U(x′, t), q(t)) solving the above problem* (6).

The relation between *q(t)* and *F(t)* depends on the solution of the inverse problem. The solvability of the time-periodic problem with the assumption that the flux *F(t)* is from the Sobolev space $W^{1,2}(0,2\pi)$ was proved in 2005. However, in applications and numerical computations usually data is not regular. Therefore, we study the problem (6) assuming **only** that $F \in L^2(0,2\pi)$.

Problem (6) can be reduced to the case when all the involved functions have zero mean values.

Let us denote by $\bar{H} = \frac{1}{2\pi} \int\limits_0^{2\pi} H(t)dt$ the mean value of a function H.

Let $(\bar{U}, \bar{q})$ be a solution of the following problem
on σ (the stationary Poiseuille solution corresponding to the flux $\bar{F}$)

$$\begin{cases} -\nu\Delta'\bar{U}(x') &= \bar{q}, \\ \bar{U}(x')|_{\partial\sigma} &= 0, \\ \int\limits_\sigma \bar{U}(x')dx' &= \bar{F}. \end{cases} \qquad (7)$$

The solution $(\bar{U}, \bar{q})$ can be represented in the form

$\bar{U}(x') = \dfrac{\bar{F}}{\kappa_0} U_0(x')$ and $\bar{q} = \dfrac{\bar{F}}{\kappa_0}$ where $U_0(x')$ is the solution of the following problem:

$$\begin{cases} -\nu\Delta' U_0(x') &= 1, \\ U_0(x')|_{\partial\sigma} &= 0, \end{cases} \quad \text{and} \quad \kappa_0 = \int\limits_\sigma U_0(x')dx' = \nu \int\limits_\sigma |\nabla' U_0(x')|^2 dx' > 0.$$

Let us represent the solution *(U, q)* in the form

$$U(x',t) = V(x',t) + \bar{U}(x'), \quad q(t) = s(t) + \bar{q}. \tag{8}$$

Then obviously, $\bar{V}(x') = 0$, $\bar{s} = 0$ and (V,s) is the solution of the problem

$$\begin{cases} V_t(x',t) - \nu\Delta' V(x',t) &= s(t), \\ V(x',t)|_{\partial\sigma} &= 0, \\ V(x',0) &= V(x',2\pi), \\ \int\limits_{\sigma} V(x',t)dx' &= \widetilde{F}(t), \end{cases} \tag{9}$$

FUNCTION SPACES

$L^2(0,T;V)$ is the Bochner space of functions u such that $u(\cdot,t) \in V$ for almost all $t \in [0,T]$ and the norm $\|u\|_{L^2(0,T;V)} = \Big(\int\limits_0^T \|u(\cdot,t)\|_V^2 dt\Big)^{\frac{1}{2}}$ is finite.

Let us consider the set of smooth periodic functions $C_\wp^\infty(0,2\pi) = \{h \in C^\infty(\mathbb{R}^1) : h(t) = h(t+2\pi)\ \forall t \in \mathbb{R}^1\}$.

Let $L^2(0,2\pi)$ be a Lebesgue space on the interval $(0,2\pi)$. We extend the functions from $L^2(0,2\pi)$ to the whole line $\mathbb{R}^1$ by putting $f(t+2\pi) = f(t)$ for any t. To emphasize that functions are periodically extended to $\mathbb{R}^1$ we use the notation $L_\wp^2(0,2\pi)$. Let $L_\sharp^2(0,2\pi) = \{h \in L_\wp^2(0,2\pi) : \int\limits_0^{2\pi} h(t)dt = 0\}$.

Denote by $W_\wp^{1,2}(0,2\pi)$ be the closure of the set $C_\wp^\infty(0,2\pi)$ in $W^{1,2}$-norm. Let $W_\wp^{-1,2}(0,2\pi)$ be dual of $W_\wp^{1,2}(0,2\pi)$, i.e., $W_\wp^{-1,2}(0,2\pi) = (W_\wp^{1,2}(0,2\pi))^*$.

PRIMITIVE FUNCTION AND ITS PROPERTIES

For any function $f \in L^2_\wp(0,2\pi)$ denote by $S_f(t)$ its primitive:

$$S_f(t) = -\int_t^{t_0+2\pi} f(\tau)d\tau, \quad \text{where } t_0 \in [0,2\pi), t \in [t_0, t_0+2\pi].$$

Clearly, $\dfrac{dS_f(t)}{dt} = f(t), \;\; S_f(t_0+2\pi) = 0.$

Moreover,

$$\int_0^{2\pi} |S_f(t)|^2 dt \leqslant 2\pi \int_0^{2\pi}\int_t^{t_0+2\pi} |f(\tau)|^2 d\tau dt \leqslant 4\pi^2 \int_{t_0}^{t_0+2\pi} |f(\tau)|^2 d\tau = 4\pi^2 \int_0^{2\pi} |f(\tau)|^2 d\tau,$$

and $S_f(t)$ is a periodic function:

$$S_f(t+2\pi) = -\int_{t+2\pi}^{t_0+2\pi} f(\tau)d\tau = -\int_t^{t_0} f(\tau)d\tau = -\int_t^{t_0+2\pi} f(\tau)d\tau + \int_{t_0}^{t_0+2\pi} f(\tau)d\tau$$
$$= S_f(t) - S_f(t_0) = S_f(t).$$

Thus, $S_f \in L^2_\wp(0,2\pi)$

DEFINITION OF A WEAK SOLUTION

Let $F \in L^2_\sharp(0,2\pi)$*. By a weak solution*

of the problem (9) *we understand a pair* (V,s) *such that*

$V \in L^2_\sharp(0,2\pi;L^2(\sigma))$*.* $s \in W^{-1,2}_\wp(0,2\pi)$*,* $V(x',t)$ *satisfies the flux condition*

$$\int\limits_\sigma V(x',t)dx' = F(t)$$

and the pair (V,s) satisfies the integral identity

$$\int\limits_0^{2\pi}\int\limits_\sigma V(x',t)\eta_t(x',t)dx'dt + \nu\int\limits_0^{2\pi}\int\limits_\sigma \nabla' S_V(x',t)\cdot\nabla'\eta_t(x',t)dx'dt$$
$$= \int\limits_0^{2\pi} S_s(t)\int\limits_\sigma \eta_t(x',t)dx'dt$$

for any test function $\eta \in L^2_\wp(0,2\pi;\mathring{W}^{1,2}(\sigma))$ such that $\eta_t \in L^2_\sharp(0,2\pi;W^{1,2}(\sigma))$

MAIN RESULT

Theorem *Let* $F \in L^2_\sharp(0,2\pi)$. *Then the problem* (9) *admits a unique weak solution* (V,s). *There holds the estimate*

$$\int_0^{2\pi}\int_\sigma |V(x',t)|^2 dx'\,dt + \int_0^{2\pi}\int_\sigma \left|\nabla' S_V(x',t)\right|^2 dx'\,dt + \int_0^{2\pi} |S_s(\tau)|^2 d\tau \leqslant c\int_0^{2\pi} |F(\tau)|^2 d\tau,$$

where the constant c *depends only on* σ.

This theorem is proved applying some version of Galerkin approximations.

CONSTRUCTION OF APPROXIMATE SOLUTION

Let $u_k(x') \in \mathring{W}^{1,2}(\sigma)$ and λ_k be eigenfunctions and eigenvalues of the Laplace operator:

$$\begin{cases} -\nu\Delta' u_k(x') &= \lambda_k u_k(x'), \\ u_k(x')\big|_{\partial\sigma} &= 0. \end{cases}$$

Note that $\lambda_k > 0$ and $\lim\limits_{k\to\infty} \lambda_k = \infty$. The eigenfunctions $u_k(x')$ are orthogonal in $L^2(\sigma)$ and we assume that $u_k(x')$ are normalized in $L^2(\sigma)$. Then

$$\nu \int\limits_\sigma |\nabla' u_k(x')|^2\, dx' = \lambda_k, \qquad \int\limits_\sigma \nabla' u_k(x')\cdot\nabla' u_l(x')\, dx' = 0, \quad k \neq l.$$

Moreover, $\{u_k(x')\}$ is a basis in $L^2(\sigma)$ and $\mathring{W}^{1,2}(\sigma)$

We look for an approximate solution of the problem (9) in the form

$$V^{(N)}(x',t) = \sum_{k=1}^{N} w_k^{(N)}(t) u_k(x').$$

$$\int\limits_\sigma V_t^{(N)}(x',t)u_k(x')dx' + \nu\int\limits_\sigma \nabla' V^{(N)}(x',t)\cdot\nabla' u_k(x')dx'$$
$$= s^{(N)}(t)\int\limits_\sigma u_k(x')dx', \quad k = 1,2,\ldots,N,$$
$$w_k^{(N)}(0) = w_k^{(N)}(2\pi), \quad k = 1,\ldots,N,$$
$$\int\limits_\sigma V^{(N)}(x',t)\,dx' = F(t),$$

$$w_k^{(N)}(t) = \beta_k \int\limits_0^{2\pi} G_k(t,\tau)s^{(N)}(\tau)d\tau,$$

$$\beta_k = \int\limits_\sigma u_k(x')dx'$$

Green function

CONSTRUCTION OF APPROXIMATE SOLUTION

$$\int_\sigma V_t^{(N)}(x',t)u_k(x')dx' + \nu \int_\sigma \nabla' V^{(N)}(x',t) \cdot \nabla' u_k(x')dx'$$
$$= s^{(N)}(t) \int_\sigma u_k(x')dx', \quad k = 1, 2, \ldots, N,$$
$$w_k^{(N)}(0) = w_k^{(N)}(2\pi), \quad k = 1, \ldots, N,$$
$$\int_\sigma V^{(N)}(x',t)\, dx' = F(t),$$

$$\xrightarrow{\text{Orthonormality of } u_k(x')}$$

$$\begin{cases} (w_k^{(N)}(t))' + \lambda_k w_k^{(N)}(t) &= \beta_k s^{(N)}(t), \quad t \in (0, 2\pi), \\ w_k^{(N)}(0) &= w_k^{(N)}(2\pi), \end{cases}$$

CONSTRUCTION OF APPROXIMATE SOLUTION

Green function

$$G_k(t,\tau) = \begin{cases} \dfrac{e^{-\lambda_k(t-\tau)}}{1-e^{-2\pi\lambda_k}}, & 0 \leqslant \tau \leqslant t \leqslant 2\pi, \\ \dfrac{e^{-\lambda_k(t-\tau+2\pi)}}{1-e^{-2\pi\lambda_k}}, & 0 \leqslant t \leqslant \tau \leqslant 2\pi. \end{cases}$$

Now the flux condition yields

$$F(t) = \int_\sigma V^{(N)}(x',t)dx' = \sum_{k=1}^{N} \beta_k \int_0^{2\pi} G_k(t,\tau)s^{(N)}(\tau)d\tau \int_\sigma u_k(x')dx'$$
$$= \sum_{k=1}^{N} \beta_k^2 \int_0^{2\pi} G_k(t,\tau)s^{(N)}(\tau)d\tau.$$

Thus for the function $s^{(N)}$ we derived Fredholm integral equation of the first kind:

$$\int_0^{2\pi} \sum_{k=1}^{N} \beta_k^2 G_k(t,\tau)s^{(N)}(\tau)d\tau = F(t).$$

It is well known that such equations, in general, are ill-posed in L^2 setting. In order to regularize the equation, we consider the following Fredholm integral equation of the second kind:

$$\alpha s_\alpha^{(N)}(t) + \int_0^{2\pi} \sum_{k=1}^{N} \beta_k^2 G_k(t,\tau)s_\alpha^{(N)}(\tau)d\tau = F(t),$$

where later α will tend to 0

CONSTRUCTION OF APPROXIMATE SOLUTION

we study the regularized problem

$$\int_{\sigma} (V_{\alpha}^{(N)})_t(x',t) u_k(x') dx' + \nu \int_{\sigma} \nabla' \underline{V_{\alpha}^{(N)}(x',t)} \cdot \nabla' u_k(x') dx'$$

$$= \underline{s_{\alpha}^{(N)}(t)} \int_{\sigma} u_k(x') dx', \quad k = 1, 2, \dots, N,$$

$$V_{\alpha}^{(N)}(x',0) = V_{\alpha}^{(N)}(x',2\pi), \tag{10}$$

$$\alpha s_{\alpha}^{(N)}(t) + \int_0^{2\pi} \sum_{k=1}^{N} \beta_k^2 G_k(t,\tau) s_{\alpha}^{(N)}(\tau) d\tau = F(t),$$

SOLUTION

the pair $(V_{\alpha}^{(N)}(x',t), s_{\alpha}^{(N)}(t))$

where

$$V_{\alpha}^{(N)}(x',t) = \sum_{k=1}^{N} w_{k,\alpha}^{(N)}(t) u_k(x'),$$

$$w_{k,\alpha}^{(N)}(t) = \beta_k \int_0^{2\pi} G_k(t,\tau) s_{\alpha}^{(N)}(\tau) d\tau,$$

A PRIORI ESTIMATES

Let the pair $(V_\alpha^{(N)}(x',t), s_\alpha^{(N)}(t))$ be the solution of the problem (10) and $U_0(x')$ be the solution of problem
$$\begin{cases} -\nu\Delta' U_0(x') &= 1, \\ U_0(x')|_{\partial\sigma} &= 0, \end{cases}$$

Consider the integral $\int_\sigma V_\alpha^{(N)}(x',t)U_0(x')dx'$. Since the mean value $\bar{V}_\alpha^{(N)}(x') = 0$, we have

$$\int_0^{2\pi}\int_\sigma V_\alpha^{(N)}(x',t)U_0(x')dx'dt = \int_\sigma U_0(x')\Big(\int_0^{2\pi} V_\alpha^{(N)}(x',t)dt\Big)dx' = 0.$$

Therefore, by the Mean Value Theorem there exists $t_* = t_*(\alpha, N)$ such that $\int_\sigma V_\alpha^{(N)}(x',t_*)U_0(x')dx' = 0$. The point $t_*(\alpha,N)$ depends on α and N

By periodicity we also have $\int_\sigma V_\alpha^{(N)}(x',t_*+2\pi)U_0(x')dx' = 0$.

Let $f \in L^2_\sharp(0,2\pi)$. For $t \in [t_*, t_*+2\pi]$ define the notation

$S_f^*(t) = -\int_t^{t_*+2\pi} f(\tau)d\tau$. Since the mean value of f vanishes, we have $S_f^*(t_*+2\pi) = S_f^*(t_*) = 0$. Moreover, $\dfrac{dS_f^*(t)}{dt} = f(t)$.

$$\int_{t_*}^{t_*+2\pi}\int_\sigma |V_\alpha^{(N)}(x',t)|^2 dx'dt \leqslant \varepsilon \int_{t_*}^{t_*+2\pi} |S^*_{s_\alpha^{(N)}}(t)|^2 dt + \frac{1}{2\varepsilon}\int_{t_*}^{t_*+2\pi} |F(t)|^2 dt,$$

$$\frac{\nu}{2}\int_{t_*}^{t_*+2\pi}\int_\sigma |\nabla' S^*_{V_\alpha^{(N)}}(x',t)|^2 dx'dt \tag{13}$$
$$\leqslant (4\pi^2+1)\Big(\varepsilon\int_{t_*}^{t_*+2\pi} |S^*_{s_\alpha^{(N)}}(t)|^2 dt + \frac{1}{2\varepsilon}\int_{t_*}^{t_*+2\pi} |F(t)|^2 dt\Big)$$

A PRIORI ESTIMATES

Let us estimate the integral $\int\limits_{t_*}^{t_*+2\pi} |S^*_{s_\alpha^{(N)}}(t)|^2 dt$. Let $U_0 \in \mathring{W}^{1,2}(\sigma)$ be a solution of the problem

$$\begin{cases} -\nu\Delta' U_0(x') & = & 1, \\ U_0(x')|_{\partial\sigma} & = & 0, \end{cases} \qquad (11)$$

Remind that the flux of U_0 is nonzero,

$$\kappa_0 = \int\limits_\sigma U_0(x')dx' > 0$$

Since $\{u_k(x')\}$ is a basis in $\mathring{W}^{1,2}(\sigma)$, U_0

can be expressed as a Fourier series in $\mathring{W}^{1,2}(\sigma)$:

$$U_0(x') = \sum_{k=1}^{\infty} a_k u_k(x'), \quad a_k \in \mathbb{R}^1.$$

A PRIORI ESTIMATES

Let us multiply the relations (10) by a_k and sum them over k. This gives

$$\int_\sigma (V_\alpha^{(N)})_t(x',t)U_0(x')dx' + \nu\int_\sigma \nabla' V_\alpha^{(N)}(x',t)\cdot\nabla' U_0(x')dx'$$
$$= s_\alpha^{(N)}(t)\int_\sigma U_0(x')dx' = s_\alpha^{(N)}(t)\kappa_0.$$

On the other hand, multiplying (11) by $V_\alpha^{(N)}(x',t)$ and integrating by parts in σ we obtain

$$\nu\int_\sigma \nabla' U_0(x')\cdot\nabla' V_\alpha^{(N)}(x',t)dx'$$
$$= \int_\sigma V_\alpha^{(N)}(x',t)dx' = F(t) - \alpha s_\alpha^{(N)}(t).$$

$$\int_\sigma (V_\alpha^{(N)})_t(x',t)U_0(x')dx' + F(t) - \alpha s_\alpha^{(N)}(t) = s_\alpha^{(N)}(t)\kappa_0,$$

i.e.,

$$(\kappa_0+\alpha)s_\alpha^{(N)}(t) = \int_\sigma (V_\alpha^{(N)})_t(x',t)U_0(x')dx' + F(t).$$

Integrating with respect to t from τ to $t_*+2\pi$ we obtain

$$(\kappa_0+\alpha)\int_\tau^{t_*+2\pi} s_\alpha^{(N)}(t)dt = -(\kappa_0+\alpha)S^*_{s_\alpha^{(N)}}(\tau)$$
$$= -\int_\sigma V_\alpha^{(N)}(x',\tau)U_0(x')dx' + \int_\tau^{t_*+2\pi} F(t)dt. \qquad (12)$$

A PRIORI ESTIMATES

Here we have used the choice of the point t_*, that is

$$\int_\sigma V_\alpha^{(N)}(x',t_*)U_0(x')dx' = \int_\sigma V_\alpha^{(N)}(x',t_*+2\pi)U_0(x')dx' = 0.$$

and hence,

$$\int_\tau^{t_*+2\pi}\int_\sigma \left(V_\alpha^{(N)}\right)_t(x',t)U_0(x')dx'dt = -\int_\sigma V_\alpha^{(N)}(x',\tau)U_0(x')dx'.$$

A PRIORI ESTIMATES

From (12) it follows that

$$(\kappa_0+\alpha)^2 \int\limits_{t_*}^{t_*+2\pi} \left|S^*_{s_\alpha^{(N)}}(\tau)\right|^2 d\tau \qquad (14)$$

$$\leqslant c\Big(\int\limits_{t_*}^{t_*+2\pi} \int\limits_{\sigma} |V_\alpha^{(N)}(x',\tau)|^2 dx'\, d\tau + \int\limits_{t_*}^{t_*+2\pi} |F(\tau)|^2 d\tau \Big).$$

Earlier we had that

$$\int\limits_{t_*}^{t_*+2\pi} \int\limits_{\sigma} |V_\alpha^{(N)}(x',t)|^2 dx'\, dt \leqslant \varepsilon \int\limits_{t_*}^{t_*+2\pi} \left|S^*_{s_\alpha^{(N)}}(t)\right|^2 dt + \frac{1}{2\varepsilon} \int\limits_{t_*}^{t_*+2\pi} |F(t)|^2 dt,$$

$$\int\limits_{t_*}^{t_*+2\pi} \int\limits_{\sigma} \left|V_\alpha^{(N)}(x',t)\right|^2 dx'\, dt \leqslant c\varepsilon \int\limits_{t_*}^{t_*+2\pi} \int\limits_{\sigma} \left|V_\alpha^{(N)}(x',t)\right|^2 dx'\, dt + c_\varepsilon \int\limits_{t_*}^{t_*+2\pi} \left|F(t)\right|^2 dt$$

and choosing ε sufficiently small we obtain

$$\int\limits_{t_*}^{t_*+2\pi} \int\limits_{\sigma} \left|V_\alpha^{(N)}(x',t)\right|^2 dx'\, dt \leqslant c \int\limits_{t_*}^{t_*+2\pi} \left|F(t)\right|^2 dt. \qquad (15)$$

A PRIORI ESTIMATES

The estimates (14) and (15) give

$$\int_{t_*}^{t_*+2\pi} \left|S^*_{s_\alpha^{(N)}}(\tau)\right|^2 d\tau \leqslant c \int_{t_*}^{t_*+2\pi} \left|F(\tau)\right|^2 d\tau. \tag{16}$$

Finally from (13) and (16) it follows that

$$\int_{t_*}^{t_*+2\pi} \int_{\sigma} \left|\nabla' S^*_{V_\alpha^{(N)}}(x',t)\right|^2 dx' dt \leqslant c \int_{t_*}^{t_*+2\pi} \left|F(t)\right|^2 dt.$$

The constants in are independent of α and N.

EXISTENCE

The approximate solution satisfies the integral identity

$$\int_0^{2\pi}\int_\sigma V_\alpha^{(N)}(x',t)\eta_t(x',t)dx'dt$$

$$+\nu\int_0^{2\pi}\int_\sigma \nabla' S^*_{V_\alpha^{(N)}}(x',t)\cdot\nabla'\eta_t(x',t)dx'dt \qquad (17)$$

$$=\int_0^{2\pi} S^*_{s_\alpha^{(N)}}(\tau)\int_\sigma \eta_t(x',t)dx'dt$$

for test functions η having the form $\eta(x',t)=\sum_{k=1}^{M} d_k(t)u_k(x')$.

$d_k(t)\in L^2_{\wp}(0,2\pi)$ such that $d_k'(t)\in L^2_{\sharp}(0,2\pi)$,

$(V_\alpha^{(N)}(x',t),s_\alpha^{(N)}(t))$ obey the a priori estimates with a constant c independent of α and N.

$$\int_0^{2\pi}\int_\sigma |V_\alpha^{(N)}(x',t)|^2dx'dt+\int_0^{2\pi}\int_\sigma \left|\nabla' S^*_{V_\alpha^{(N)}}(x',t)\right|^2dx'dt+$$

$$\int_0^{2\pi}|S^*_{s_\alpha^{(N)}}(\tau)|^2d\tau\leqslant c\int_0^{2\pi}|F(\tau)|^2d\tau.$$

EXISTENCE

Let us fix N and choose a subsequences $\{\alpha_l\}$ and $\{(V_{\alpha_l}^{(N)}(x',t), s_{\alpha_l}^{(N)}(t))\}$ such that $\lim\limits_{l\to\infty} \alpha_l = 0$, $\{V_{\alpha_l}^{(N)}\}$ converges weakly in $L^2_\sharp(0,2\pi;L^2(\sigma))$ to some $V^{(N)}$, $\left\{S^*_{V_{\alpha_l}^{(N)}}\right\}$ converges weakly in $L^2_\wp(0,2\pi;\mathring{W}^{1,2}(\sigma))$ to $S_{V^{(N)}}$. Recall that for $U \in L^2_\sharp(0,T;L^2(\sigma))$, and S_U is the primitive of U. Moreover, $\left\{s_{\alpha_l}^{(N)}\right\}$ converges weakly in $W_\wp^{-1,2}(0,2\pi)$ to $s^{(N)}$. The last convergence means that

$$\lim_{l\to\infty} \int_0^{2\pi} S^*_{s_{\alpha_l}^{(N)}}(t)\eta'(t)dt = \int_0^{2\pi} S_{s^{(N)}}(t)\eta'(t)dt = \langle s^{(N)}, \eta\rangle \quad \forall \eta \in W_\wp^{1,2}(0,2\pi),$$

In (17) taking $\alpha = \alpha_l$ and passing to the limit as $\alpha_l \to 0$, we get

$$\int_0^{2\pi}\int_\sigma V^{(N)}(x',t)\eta_t(x',t)dx'dt + \nu \int_0^{2\pi}\int_\sigma \nabla' S_{V^{(N)}}(x',t)\cdot\nabla'\eta_t(x',t)dx'dt = \int_0^{2\pi} S_{s^{(N)}}(\tau)\int_\sigma \eta_t(x',t)dx'dt. \tag{19}$$

Obviously, for the limit functions $V^{(N)}$ and $S_{s^{(N)}}$ remain valid with a constant c independent of N.

EXISTENCE

Let us show that $V^{(N)}(x',t)$ satisfy the flux condition:

$$\int\limits_{\sigma} V^{(N)}(x',t)dx' = F(t).$$

Integrating the equation

$$\alpha s_{\alpha}^{(N)}(t) + \int\limits_{0}^{2\pi} \sum_{k=1}^{N} \beta_k^2 G_k(t,\tau) s_{\alpha}^{(N)}(\tau)d\tau = F(t),$$

for $\alpha = \alpha_l$ from t to 2π yields

$$\alpha_l S_{s_{\alpha_l}^{(N)}}(t) + \int\limits_{t}^{2\pi} \int\limits_{\sigma} V_{\alpha_l}^{(N)}(x',\tau)dx'd\tau = S_F(t). \qquad (18)$$

EXISTENCE

Obviously, the sequence $\left\{\varphi_l^{(N)}(\tau) = \int\limits_\sigma V_{\alpha_l}^{(N)}(x',\tau)dx'\right\}$ is bounded in $L^2(0,2\pi)$. So we may assume, without loss of generality, that $\left\{\varphi_l^{(N)}(\tau)\right\}$ is weakly convergent to $\varphi^{(N)}$ in $L^2(0,2\pi)$. Then, the sequence of primitives $S_{\varphi_l^{(N)}}(t) = \int\limits_t^{2\pi} \varphi_l^{(N)}(\tau)d\tau \to S_{\varphi^{(N)}}(t)$ for all $t \in [0,2\pi]$ and hence $\|S_{\varphi_l^{(N)}} - S_{\varphi^{(N)}}\|_{L^2(0,2\pi)} \to 0$ as $l \to \infty$ $(\alpha_l \to 0)$. From (18) we have

$$\|S_{\varphi_l^{(N)}} - S_F\|_{L^2(0,2\pi)} = \alpha_l \|S_{s_{\alpha_l}^{(N)}}\|_{L^2(0,2\pi)} \le c\alpha_l \to 0 \quad \text{as} \quad l \to \infty.$$

Therefore,

$$\int\limits_t^{2\pi}\int\limits_\sigma V^{(N)}(x',\tau)dx'd\tau = \int\limits_t^{2\pi} F(\tau)d\tau \quad \text{for a.a.} \quad t \in [0,2\pi],$$

and differentiating this equality with respect to t we get the flux condition.

EXISTENCE

Since the pair $\left(V^{(N)}(x',t),\, s^{(N)}(t)\right)$ obeys the same a priori estimates with the constants independent of N, there exists a subsequence $\{\left(V^{(N_k)}(x',t),\, s^{(N_k)}(t)\right)\}$ such that $\{V^{(N_k)}\}$ converges weakly in $L^2_\sharp(0,2\pi;L^2(\sigma))$ to some V, $\left\{S_{V^{(N_k)}}\right\}$ converges weakly in $L^2_\wp(0,2\pi;\mathring{W}^{1,2}(\sigma))$ to S_V and $\{s^{(N_k)}\}$ converges weakly in $W_\wp^{-1,2}(0,2\pi)$ to s. In (19) passing to the limit as $N_k \to +\infty$, we obtain

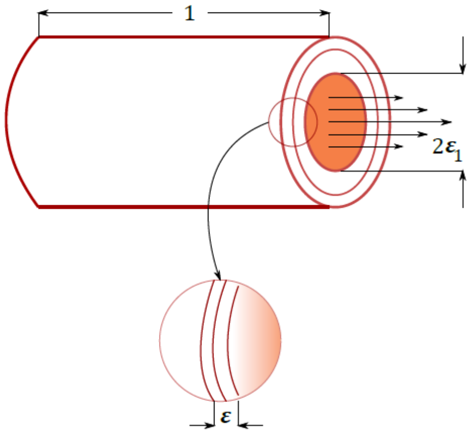
$$\int_0^{2\pi}\int_\sigma V(x',t)\eta_t(x',t)dx'dt + \nu\int_0^{2\pi}\int_\sigma \nabla' S_V(x',t)\cdot\nabla'\eta_t(x',t)dx'dt$$
$$= \int_0^{2\pi} S_s(\tau)\int_\sigma \eta_t(x',t)dx'dt$$

Integral identity is proved for test functions η which can be represented as the sums: $\eta(x',t)=\sum\limits_{k=1}^{M} d_k(t)u_k(x')$ with $d_k(t)\in L^2_\wp(0,2\pi)$ such that $d_k'(t)\in L^2_\sharp(0,2\pi)$.
But such sums are dense in the space of test functions. Therefore, remains valid for all the test functions η.
Moreover, $V(x',t)$ satisfies the flux condition:

$$\int_\sigma V(x',t)dx' = F(t).$$

Poiseuille-type approximations for axisymmetric flow in a thin tube with thin stiff elastic wall

NOTATION



$\varepsilon << \varepsilon_1 << 1$

$$C^f = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < \varepsilon_1^2, \, x_3 \in \mathbb{R}\}$$

$$C_\varepsilon^e = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \varepsilon_1^2 < x_1^2 + x_2^2 < (\varepsilon_1 + \varepsilon)^2, \, x_3 \in \mathbb{R}\}$$

since $\quad r = \sqrt{x_1^2 + x_2^2}$

$$L^f = \{(x_3, r) \in \mathbb{R}^2 : x_3 \in \mathbb{R}, \, r \in (0, \varepsilon_1)\}$$

$$L_\varepsilon^e = \{(x_3, r) \in \mathbb{R}^2 : x_3 \in \mathbb{R}, \, r \in (\varepsilon_1, \varepsilon_1 + \varepsilon)\}$$

and

$$F^0 = \{(x_3, 0) : x_3 \in \mathbb{R}\}$$

$$F^{\varepsilon_1} = \{(x_3, \varepsilon_1) : x_3 \in \mathbb{R}\}$$

$$F^{\varepsilon_1 + \varepsilon} = \{(x_3, \varepsilon_1 + \varepsilon) : x_3 \in \mathbb{R}\}$$

NOTATION

$$\mathbf{L}\mathbf{u}\cdot\beta_3 = \frac{\partial}{\partial x_3}\left((\lambda+2\mu)\frac{\partial u_3}{\partial x_3}+\lambda\left(\frac{\partial u_r}{\partial r}+\frac{1}{r}u_r\right)\right)+\frac{\partial}{\partial r}\left(\mu\left(\frac{\partial u_3}{\partial r}+\frac{\partial u_r}{\partial x_3}\right)\right)+\frac{\mu}{r}\left(\frac{\partial u_3}{\partial r}+\frac{\partial u_r}{\partial x_3}\right)$$

Linear elasticity operator

$$\mathbf{L}\mathbf{u}\cdot\beta_r = \frac{\partial}{\partial x_3}\left(\mu\left(\frac{\partial u_3}{\partial r}+\frac{\partial u_r}{\partial x_3}\right)\right)+\frac{\partial}{\partial r}\left(\lambda\left(\frac{\partial u_3}{\partial x_3}+\frac{1}{r}u_r\right)+(\lambda+2\mu)\frac{\partial u_r}{\partial r}\right)+\frac{2\mu}{r}\left(\frac{\partial u_r}{\partial r}-\frac{1}{r}u_r\right)$$

$$\mathrm{div}_c\mathbf{u} = \frac{\partial u_3}{\partial x_3}+\frac{\partial u_r}{\partial r}+\frac{1}{r}u_r$$

Divergence operator for a vector-valued function

$$\mathrm{div}_c S = \left(\frac{\partial S_{33}}{\partial x_3}+\frac{1}{r}\frac{\partial}{\partial r}(rS_{r3})\right)\beta_3 + \left(\frac{\partial S_{3r}}{\partial x_3}+\frac{1}{r}\frac{\partial}{\partial r}(rS_{rr})-\frac{S_{\theta\theta}}{r}\right)\beta_r$$

Divergence operator for a symmetric tensor-valued function

$$\nabla_c\mathbf{u} = \begin{pmatrix} \frac{\partial u_3}{\partial x_3} & 0 & \frac{\partial u_3}{\partial r} \\ 0 & \frac{1}{r}u_r & 0 \\ \frac{\partial u_r}{\partial x_3} & 0 & \frac{\partial u_r}{\partial r} \end{pmatrix}$$

The gradient operator for a vector-valued function

$$D_c(\mathbf{u}) = \frac{1}{2}\left(\nabla_c\mathbf{u}+(\nabla_c\mathbf{u})^T\right)$$

Velocity strain tensor

RESULT OF G. PANASENKO AND R. STAVRE (2020)

$$v_3(x_3,r,t)=4\varepsilon_1^2\left(1-\frac{r^2}{\varepsilon_1^2}\right)Q(x_3,t)$$
$$+\frac{\partial w_3}{\partial t}(x_3,t)+\frac{\varepsilon_1^2}{4}\left(1-\frac{r^2}{\varepsilon_1^2}\right)\left(-\frac{\rho_f}{\nu}\frac{\partial^2 w_3}{\partial t^2}(x_3,t)+\frac{\partial^3 w_3}{\partial t\partial x_3^2}(x_3,t)\right),$$
$$v_r(x_3,r,t)=-\varepsilon_1^3\frac{r}{\varepsilon_1}\left(2-\frac{r^2}{\varepsilon_1^2}\right)\frac{\partial Q}{\partial x_3}(x_3,t)-\varepsilon_1\frac{r}{2\varepsilon_1}\frac{\partial^2 w_3}{\partial t\partial x_3}(x_3,t)$$
$$-\frac{\varepsilon_1^3}{16}\frac{r}{\varepsilon_1}\left(2-\frac{r^2}{\varepsilon_1^2}\right)\left(-\frac{\rho_f}{\nu}\frac{\partial^3 w_3}{\partial t^2\partial x_3}(x_3,t)+\frac{\partial^4 w_3}{\partial t\partial x_3^3}(x_3,t)\right),$$
$$p(x_3,r,t)=q(x_3,t),$$
$$u_3(x_3,r,t)=w_3(x_3,t)+\varepsilon\frac{r-\varepsilon_1}{\varepsilon}\left(\varepsilon_1^3\int_0^t\frac{\partial^2 Q}{\partial x_3^2}(x_3,\theta)\mathrm{d}\theta\right.$$
$$\left.+\frac{\varepsilon_1}{2}\frac{\partial^2 w_3}{\partial x_3^2}(x_3,t)\right)-\nu\omega_E^{-1}\varepsilon\varepsilon_1\left(\int_0^{\frac{r-\varepsilon_1}{\varepsilon}}\frac{1-\tau}{\mu(\tau)}\mathrm{d}\tau\right)$$
$$\times\left(8Q(x_3,t)-\frac{\rho_f}{2\nu}\frac{\partial^2 w_3}{\partial t^2}(x_3,t)+\frac{\partial^3 w_3}{\partial t\partial x_3^2}(x_3,t)\right),$$

$$u_r(x_3,r,t)=-\varepsilon_1^3\left(1-\varepsilon\int_0^{\frac{r-\varepsilon_1}{\varepsilon}}\frac{1}{\varepsilon_1+\varepsilon\tau}\frac{\lambda(\tau)}{\lambda(\tau)+2\mu(\tau)}\mathrm{d}\tau\right)$$
$$\times\int_0^t\frac{\partial Q}{\partial x_3}(x_3,\theta)\mathrm{d}\theta-\left(\frac{\varepsilon_1}{2}\left(1-\varepsilon\int_0^{\frac{r-\varepsilon_1}{\varepsilon}}\frac{1}{\varepsilon_1+\varepsilon\tau}\frac{\lambda(\tau)}{\lambda(\tau)+2\mu(\tau)}\mathrm{d}\tau\right)\right.$$
$$\left.+\varepsilon\int_0^{\frac{r-\varepsilon_1}{\varepsilon}}\frac{\lambda(\tau)}{\lambda(\tau)+2\mu(\tau)}\mathrm{d}\tau\right)\frac{\partial w_3}{\partial x_3}(x_3,t)$$
$$+\omega_E^{-1}\varepsilon\left(\int_0^{\frac{r-\varepsilon_1}{\varepsilon}}\frac{1-\tau}{\lambda(\tau)+2\mu(\tau)}\mathrm{d}\tau\right)\left(2\nu\varepsilon_1^2\frac{\partial Q}{\partial x_3}(x_3,t)\right.$$
$$\left.-\nu\frac{\partial^2 w_3}{\partial t\partial x_3}(x_3,t)-q(x_3,t)\right).$$

Here, for the leading terms, we keep the same notation as for the exact solution.

RESULT OF G. PANASENKO AND R. STAVRE (2020)

Note that the leading term for pressure, q, is related to the scaled average velocity Q by

$$\frac{\partial q}{\partial x_3}(x_3,t) + 16\nu Q(x_3,t) = f_3,$$

where f_3 is a longitudial external force which represents action on a fluid. So, from (4.9) we can consider only two independent basic functions of the leading term of the ansatz and the radial displacement of the wall-fluid interface, w_r, can be approximately calculated as

$$w_r(x_3,t) = -\varepsilon_1^3 \int_0^t \frac{\partial Q}{\partial x_3}(x_3,\tau)\mathrm{d}\tau - \frac{\varepsilon_1}{2}\frac{\partial w_3}{\partial x_3}(x_3,t),$$

and so,

$$\frac{\partial w_r}{\partial t}(x_3,t) = -\varepsilon_1^3 \frac{\partial Q}{\partial x_3}(x_3,t) - \frac{\varepsilon_1}{2}\frac{\partial^2 w_3}{\partial t \partial x_3}(x_3,t).$$

If we need a continuous approximation of the velocity at the interface, then we have to add the third order terms in the approximation of u_r:

$$u_r(x_3,t) = -\frac{\varepsilon_1^3}{16}\left(-\frac{\rho_f}{\nu}\frac{\partial^2 w_3}{\partial t \partial x_3}(x_3,t) + \frac{\partial^3 w_3}{\partial x_3^3}(x_3,t)\right).$$

MATHEMATICAL MODEL

$$
\left\{
\begin{array}{ll}
\omega_\rho \rho_e \dfrac{\partial^2 \mathbf{u}}{\partial t^2} - \omega_E L\mathbf{u} = \varepsilon^{-1}\mathbf{g} & \text{in } L^e_\varepsilon \times (0,T), \\
\left\{
\begin{array}{l}
\rho_f \dfrac{\partial \mathbf{v}}{\partial t} - 2\nu \mathrm{div}_c D_c(\mathbf{v}) + \nabla p = \mathbf{f} \\
\mathrm{div}_c \mathbf{v} = 0
\end{array}
\right. & \text{in } L^f \times (0,T), \\
v_r = 0 & \text{on } F^0 \times (0,T), \\
\left\{
\begin{array}{l}
\dfrac{\partial u_3}{\partial r} + \dfrac{\partial u_r}{\partial x_3} = 0 \\
\lambda(1) \dfrac{\partial u_3}{\partial x_3} + (\lambda(1) + 2\mu(1)) \dfrac{\partial u_r}{\partial r} \\
+ \dfrac{\lambda(1)}{\varepsilon_1 + \varepsilon} u_r = 0
\end{array}
\right. & \text{on } F^{\varepsilon_1+\varepsilon} \times (0,T), \\
\left\{
\begin{array}{l}
\mathbf{v} = \dfrac{\partial \mathbf{u}}{\partial t} \\
\nu \left(\dfrac{\partial v_3}{\partial r} + \dfrac{\partial v_r}{\partial x_3} \right) = \omega_E \mu(0) \left(\dfrac{\partial u_3}{\partial r} + \dfrac{\partial u_r}{\partial x_3} \right) \\
-p + 2\nu \dfrac{\partial v_r}{\partial r} = \omega_E \Big(\lambda(0) \dfrac{\partial u_3}{\partial x_3} + (\lambda(0) \\
+2\mu(0)) \dfrac{\partial u_r}{\partial r} + \dfrac{\lambda(0)}{\varepsilon_1} u_r \Big)
\end{array}
\right. & \text{on } F^{\varepsilon_1} \times (0,T), \\
\mathbf{u}, \mathbf{v}, p & 1 - \text{periodic in } x_3, \\
\mathbf{u}(0) = \dfrac{\partial \mathbf{u}}{\partial t}(0) = \mathbf{0} & \text{in } L^e_\varepsilon, \\
\mathbf{v}(0) = \mathbf{0} & \text{in } L^f.
\end{array}
\right. \qquad \text{(M)}
$$

THE VARIATIONAL FRAMEWORK OF THE PROBLEM

$$\Omega^f = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < \varepsilon_1^2, x_3 \in (0,1)\}$$

$$\Omega_\varepsilon^e = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : \varepsilon_1^2 < x_1^2 + x_2^2 < (\varepsilon_1 + \varepsilon)^2, x_3 \in (0,1)\}$$

$$D^f = \{(x_3,r) \in \mathbb{R}^2 : x_3 \in (0,1), r \in (0,\varepsilon_1)\}$$

$$D_\varepsilon^e = \{(x_3,r) \in \mathbb{R}^2 : x_3 \in (0,1), r \in (\varepsilon_1, \varepsilon_1 + \varepsilon)\}$$

$$\Gamma^0 = \{(x_3,0) : x_3 \in (0,1)\}$$

$$\Gamma^{\varepsilon_1} = \{(x_3,\varepsilon_1) : x_3 \in (0,1)\}$$

$$\Gamma^{\varepsilon_1+\varepsilon} = \{(x_3,\varepsilon_1+\varepsilon) : x_3 \in (0,1)\}$$

For the fluid domain we consider the following spaces

$$L_r^2(D^f) = \{\psi : D^f \mapsto \mathbb{R}^2 : \int_{D^f} r\,\psi^2(x_3,r)\mathrm{d}x_3\mathrm{d}r < \infty\},$$

$$W_r^{1,2}(D^f) = \{\psi \in L_r^2(D^f) : \int_{D^f} r|\nabla_c\psi|^2(x_3,r)\mathrm{d}x_3\mathrm{d}r < \infty\},$$

$$\mathring{W}_r^{1,2}(D^f) = \{\psi \in W_r^{1,2}(D^f) : r\psi = \mathbf{0} \text{ on } \Gamma^{\varepsilon_1}\},$$

$$W_r^{2,2}(D^f) = \{\psi \in W_r^{1,2}(D^f) : \int_{D^f} r|\nabla_c^2\psi|^2(x_3,r)\mathrm{d}x_3\mathrm{d}r < \infty\},$$

where

$$|\nabla_c^2\psi|^2 = \left(\frac{\partial^2\psi_3}{\partial x_3^2}\right)^2 + \left(\frac{\partial^2\psi_3}{\partial r^2}\right)^2 + 2\left(\frac{\partial^2\psi_3}{\partial x_3\partial r}\right)^2 + \left(\frac{\partial^2\psi_r}{\partial x_3^2}\right)^2 + \left(\frac{\partial^2\psi_r}{\partial r^2}\right)^2$$
$$+2\left(\frac{\partial^2\psi_r}{\partial x_3\partial r}\right)^2 + \frac{1}{r^2}\left(\left(\frac{\partial\psi_3}{\partial r}\right)^2 + 2\left(\frac{\partial\psi_r}{\partial x_3}\right)^2 + 3\left(\frac{\partial\psi_r}{\partial r} - \frac{1}{r}\psi_r\right)^2\right).$$

THE VARIATIONAL FRAMEWORK OF THE PROBLEM

In the framework presented above, the variational formulation of system (M) developed by G. Panasenko and R. Stavre can be expressed as follows:

$$\begin{cases} \text{Find } (\mathbf{u},\mathbf{v}) \in H_U \times H_V, \text{ such that} \\ \omega_\rho \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_{D_\varepsilon^e} r\rho_e \dfrac{\partial \mathbf{u}(t)}{\partial t} \cdot \varphi + \omega_E a_L(\mathbf{u}(t),\varphi) + \rho_f \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_{D^f} r\mathbf{v}(t)\cdot\psi \\ +2\nu \displaystyle\int_{D^f} r D_c(\mathbf{v}(t)) : D_c(\psi) = \varepsilon^{-1} \displaystyle\int_{D_\varepsilon^e} r\mathbf{g}(t)\cdot\varphi \\ + \displaystyle\int_{D^f} r\mathbf{f}(t)\cdot\psi \quad \forall(\varphi,\psi)\in S_U, \quad \text{a.e. in } (0,T), \\ \dfrac{\partial \mathbf{u}}{\partial t} = \mathbf{v} \qquad \text{in } L^2(0,T;W_{per}^{1/2,2}(\Gamma^1)), \\ \mathbf{u}(0) = \dfrac{\partial \mathbf{u}}{\partial t}(0) = \mathbf{0} \qquad \text{in } L^2_{r,per}(D_\varepsilon^e), \\ \mathbf{v}(0) = \mathbf{0} \qquad \text{in } L^2_{r,per}(D^f), \end{cases}$$

$$U = \left\{ \varphi \in W^{1,2}_{r,per}(D^e_\varepsilon) : \int_0^1 \varphi_r(x_3,1)\mathrm{d}x_3 = 0, \right\},$$

$$V = \left\{ \psi \in W^{1,2}_{r,per}(D^f) : \mathrm{div}_c\,\psi = 0,\ \psi_r = 0 \text{ on } \Gamma^0 \right\},$$

$$H_U = \left\{ \varphi \in W^{1,2}(0,T;U) : \frac{\partial^2\varphi}{\partial t^2} \in L^2(0,T;U') \right\},$$

$$H_V = \left\{ \psi \in L^2(0,T;V) : \frac{\partial\psi}{\partial t} \in L^2(0,T;V') \right\}.$$

where a_L, defined by

$$a_L(\mathbf{u},\varphi) = \int_{D_\varepsilon^e} r\Big(\mu\Big(2\Big(\frac{\partial u_3}{\partial x_3}\frac{\partial \varphi_3}{\partial x_3} + \frac{\partial u_r}{\partial r}\frac{\partial \varphi_r}{\partial r}\Big) \\ + \Big(\frac{\partial u_3}{\partial r} + \frac{\partial u_r}{\partial x_3}\Big)\Big(\frac{\partial \varphi_3}{\partial r} + \frac{\partial \varphi_r}{\partial x_3}\Big) + 2\frac{u_r}{r}\frac{\varphi_r}{r}\Big) + \lambda \mathrm{div}_c\mathbf{u}\,\mathrm{div}_c\varphi\Big)$$

MODIFIED VARIATIONAL FORMULATION FOR NUMERICAL SETUP

We will modify the boundary conditions at the ends of the tube. Instead of the periodic solution with respect to the variable x_3 we introduce some given inflow and outflow supposing the tube with elastic wall being clamped at the ends of the tube.

$$
\begin{cases}
\omega_\rho \rho_e \dfrac{\partial^2 \mathbf{u}}{\partial t^2} - \omega_E L\mathbf{u} = 0 & \text{in } L_\varepsilon^e \times (0,T), \\
\begin{cases} \rho_f \dfrac{\partial \mathbf{v}}{\partial t} - 2\nu \operatorname{div}_c D_c(\mathbf{v}) + \nabla p = 0 \\ \operatorname{div}_c \mathbf{v} = 0 \end{cases} & \text{in } L^f \times (0,T), \\
v_r = 0 & \text{on } F^0 \times (0,T), \\
\begin{cases} \dfrac{\partial u_3}{\partial r} + \dfrac{\partial u_r}{\partial x_3} = 0 \\ \lambda(1)\dfrac{\partial u_3}{\partial x_3} + (\lambda(1)+2\mu(1))\dfrac{\partial u_r}{\partial r} + \dfrac{\lambda(1)}{\varepsilon_1+\varepsilon} u_r = 0 \end{cases} & \text{on } F^{\varepsilon_1+\varepsilon} \times (0,T), \\
\begin{cases} \mathbf{v} = \dfrac{\partial \mathbf{u}}{\partial t} \\ \nu\left(\dfrac{\partial v_3}{\partial r} + \dfrac{\partial v_r}{\partial x_3}\right) = \omega_E \mu(0)\left(\dfrac{\partial u_3}{\partial r} + \dfrac{\partial u_r}{\partial x_3}\right) \\ -p + 2\nu \dfrac{\partial v_r}{\partial r} = \omega_E\left(\lambda(0)\dfrac{\partial u_3}{\partial x_3} + (\lambda(0)+2\mu(0))\dfrac{\partial u_r}{\partial r} + \dfrac{\lambda(0)}{\varepsilon_1} u_r\right) \end{cases} & \text{on } F^{\varepsilon_1} \times (0,T), \\
v_r = \frac{1}{4\nu}(\varepsilon_1^2 - r^2) g_{in}(t), v_3 = 0, \mathbf{u} = 0 & \text{for } x_3 = 0, \\
v_r = \frac{1}{4\nu}(\varepsilon_1^2 - r^2) g_{out}(t), v_3 = 0, \mathbf{u} = 0 & \text{for } x_3 = 1, \\
\mathbf{u}(0) = \dfrac{\partial \mathbf{u}}{\partial t}(0) = \mathbf{0} & \text{in } L_\varepsilon^e, \\
\mathbf{v}(0) = \mathbf{0} & \text{in } L^f.
\end{cases}
$$

$$
\tilde{U} = \{\varphi \in W_r^{1,2}(D_\varepsilon^e)\},
$$

$$
\tilde{V} = \left\{\psi \in W_r^{1,2}(D^f) : \operatorname{div}_c \psi = 0,\ \psi_r = 0 \text{ on } \Gamma^0\right\},
$$

$$
\tilde{H}_{\tilde{U}} = \left\{\varphi \in W^{1,2}(0,T;\tilde{U}) :\ \frac{\partial^2 \varphi}{\partial t^2} \in L^2(0,T;\tilde{U}')\right\},
$$

$$
\tilde{H}_{\tilde{V}} = \left\{\psi \in L^2(0,T;\tilde{V}) :\ \frac{\partial \psi}{\partial t} \in L^2(0,T;\tilde{V}')\right\}.
$$

MODIFIED VARIATIONAL FORMULATION FOR NUMERICAL SETUP

$$\begin{cases} \text{Find } (\mathbf{u},\mathbf{v}) \in \tilde{H}_{\tilde{U}} \times \tilde{H}_{\tilde{V}}, \text{ such that} \\ v_r = \frac{1}{4\nu}(\varepsilon_1^2 - r^2) g_{in}(t), v_3 = 0, \mathbf{u} = 0 \text{ for } x_3 = 0, \\ v_r = \frac{1}{4\nu}(\varepsilon_1^2 - r^2) g_{out}(t), v_3 = 0, \mathbf{u} = 0 \text{ for } x_3 = 1, \text{ and} \\ \omega_\rho \int_0^T \int_{D_\varepsilon^e} r \rho_e \frac{\partial^2 \mathbf{u}(t)}{\partial t^2} \cdot \varphi \mathrm{d}t + \int_0^T \omega_E \, a_L(\mathbf{u}(t), \varphi) \mathrm{d}t \\ + \int_0^T \rho_f \int_{Df} r \frac{\partial}{\partial t} \mathbf{v}(t) \cdot \psi \mathrm{d}t + 2 \int_0^T \nu \int_{Df} r D_c(\mathbf{v}(t)) : D_c(\psi) \mathrm{d}t = 0 \\ \forall (\varphi, \psi) \in \tilde{H}_{\tilde{U}} \times \tilde{H}_{\tilde{V}}, \text{ such that } \varphi|_{x_3=0;1} = 0, \\ \psi|_{x_3=0;1} = 0, \text{ and} \frac{\partial \varphi}{\partial t} = \psi \qquad \text{in } L^2(0,T; W^{1/2,2}(\Gamma^1)), \\ \varphi(0) = \varphi(T) = \frac{\partial \varphi}{\partial t}(0) = \frac{\partial \varphi}{\partial t}(T) = \mathbf{0} \qquad \text{in } L_r^2(D_\varepsilon^e), \\ \psi(0) = \psi(T) = \mathbf{0} \qquad \text{in } L_r^2(D^f). \end{cases}$$

Here $2\varepsilon^2 Q$ is the average velocity, $2\pi\varepsilon^4 Q$ is the flux.

$$\omega_\rho \int_0^T \int_{D_\varepsilon^e} r \rho_e \Big(\frac{\partial^2 u_3}{\partial t^2} \varphi_3 + \frac{\partial^2 u_r}{\partial t^2} \varphi_r \Big)$$

$$+\omega_E \int_0^T \int_{D_\varepsilon^e} \Big(2\mu r \Big(\frac{\partial u_3}{\partial x_3} \frac{\partial \varphi_3}{\partial x_3} + \frac{\partial u_r}{\partial r} \frac{\partial \varphi_r}{\partial r} \Big)$$

$$+\mu r \Big(\frac{\partial u_3}{\partial r} + \frac{\partial u_r}{\partial x_3} \Big) \Big(\frac{\partial \varphi_3}{\partial r} + \frac{\partial \varphi_r}{\partial x_3} \Big) + 2\mu r \frac{u_r}{r} \frac{\varphi_r}{r}$$

$$+\lambda r \Big(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r} \Big) \Big(\frac{\partial \varphi_3}{\partial x_3} + \frac{\partial \varphi_r}{\partial r} + \frac{\varphi_r}{r} \Big) \Big)$$

$$+\rho_f \int_0^T \int_{Df} r \Big(\frac{\partial v_3}{\partial t} \psi_3 + \frac{\partial v_r}{\partial t} \psi_r \Big)$$

$$+2\nu \int_0^T \int_{Df} r \Big(\frac{\partial v_3}{\partial x_3} \frac{\partial \psi_3}{\partial x_3} + \frac{1}{2} \Big(\frac{\partial v_3}{\partial r} + \frac{\partial v_r}{\partial x_3} \Big) \Big(\frac{\partial \psi_3}{\partial r} + \frac{\partial \psi_r}{\partial x_3} \Big)$$

MODIFIED VARIATIONAL FORMULATION FOR NUMERICAL SETUP

$$v_3(x_3,r,t)=4\varepsilon_1^2\left(1-\frac{r^2}{\varepsilon_1^2}\right)Q(x_3,t),$$

$$v_r(x_3,r,t)=-\varepsilon_1^3\frac{r}{\varepsilon_1}\left(2-\frac{r^2}{\varepsilon_1^2}\right)\frac{\partial Q}{\partial x_3}(x_3,t),$$

$$u_3(x_3,r,t)=\varepsilon\frac{r-\varepsilon_1}{\varepsilon}\left(\varepsilon_1^3\int\limits_0^t\frac{\partial^2 Q}{\partial x_3^2}(x_3,\theta)\mathrm{d}\theta\right)$$

$$-8\nu\omega_E^{-1}\varepsilon\varepsilon_1\left(\int\limits_0^{\frac{r-\varepsilon_1}{\varepsilon}}\frac{1-\tau}{\mu(\tau)}\mathrm{d}\tau\right)Q(x_3,t),$$

$$u_r(x_3,r,t)=-\varepsilon_1^3\left(1-\varepsilon\int\limits_0^{\frac{r-\varepsilon_1}{\varepsilon}}\frac{1}{\varepsilon_1+\varepsilon\tau}\frac{\lambda(\tau)}{\lambda(\tau)+2\mu(\tau)}\mathrm{d}\tau\right)$$

$$\times\int\limits_0^t\frac{\partial Q}{\partial x_3}(x_3,\theta)\mathrm{d}\theta+\omega_E^{-1}\varepsilon\left(\int\limits_0^{\frac{r-\varepsilon_1}{\varepsilon}}\frac{1-\tau}{\lambda(\tau)+2\mu(\tau)}\mathrm{d}\tau\right)$$

$$\times\left(2\nu\varepsilon_1^2\frac{\partial Q}{\partial x_3}(x_3,t)+16\nu\int\limits_0^{x_3}Q(s,t)\mathrm{d}s\right),$$

$$\longrightarrow$$

$$\tilde{C}_1\frac{\partial^4 Q(x_3,t)}{\partial x_3^4}+\tilde{C}_2\frac{\partial^2 Q(x_3,t)}{\partial t^2}+\tilde{C}_3\frac{\partial^2 Q(x_3,t)}{\partial x_3^2}+\tilde{C}_4\frac{\partial^4 Q(x_3,t)}{\partial x_3^2\partial t^2}$$

$$+\tilde{C}_5\int\limits_0^{x_3}\int\limits_0^{\theta}\frac{\partial^2 Q(s,t)}{\partial t^2}\mathrm{d}s\mathrm{d}\theta+\tilde{C}_6\int\limits_0^{t}\int\limits_0^{\tau}\frac{\partial^6 Q(x_3,\theta)}{\partial x_3^6}\mathrm{d}\theta\mathrm{d}\tau$$

$$+\tilde{C}_7\int\limits_0^{t}\int\limits_0^{\tau}\frac{\partial^2 Q(x_3,\theta)}{\partial x_3^2}\mathrm{d}\theta\mathrm{d}\tau+\tilde{C}_8 Q(x_3,t)+\tilde{C}_9\int\limits_0^{x_3}\int\limits_0^{\theta}Q(s,t)\mathrm{d}s\mathrm{d}\theta$$

$$+\tilde{C}_{10}\int\limits_0^{t}\int\limits_0^{\tau}\frac{\partial^4 Q(x_3,\theta)}{\partial x_3^4}\mathrm{d}\theta\mathrm{d}\tau+\tilde{C}_{11}\frac{\partial Q(x_3,t)}{\partial t}+\tilde{C}_{12}\frac{\partial^3 Q(x_3,t)}{\partial x_3^2\partial t}=0,$$

MODIFIED VARIATIONAL FORMULATION FOR NUMERICAL SETUP

Further we will consider a shorter approximation for the solution:

$$v_3(x_3,r,t) = 4\varepsilon_1^2\left(1-\frac{r^2}{\varepsilon_1^2}\right)Q(x_3,t),$$
$$v_r(x_3,r,t) = -\varepsilon_1^3\frac{r}{\varepsilon_1}\left(2-\frac{r^2}{\varepsilon_1^2}\right)\frac{\partial Q}{\partial x_3}(x_3,t),$$
$$u_3(x_3,r,t) = -8\nu\omega_E^{-1}\varepsilon\varepsilon_1\left(\int_0^{\frac{r-\varepsilon_1}{\varepsilon}}\frac{1-\tau}{\mu(\tau)}\mathrm{d}\tau\right)Q(x_3,t),$$
$$u_r(x_3,r,t) = 2\omega_E^{-1}\varepsilon\nu\left(\int_0^{\frac{r-\varepsilon_1}{\varepsilon}}\frac{1-\tau}{\lambda(\tau)+2\mu(\tau)}\mathrm{d}\tau\right)\varepsilon_1^2\frac{\partial Q}{\partial x_3}(x_3,t).$$

We assume that μ and λ are constants, so we have the following expressions:

$$v_3(x_3,r,t) = 4\varepsilon_1^2\left(1-\frac{r^2}{\varepsilon_1^2}\right)Q(x_3,t),$$
$$v_r(x_3,r,t) = -\varepsilon_1^3\frac{r}{\varepsilon_1}\left(2-\frac{r^2}{\varepsilon_1^2}\right)\frac{\partial Q}{\partial x_3}(x_3,t), \tag{N}$$
$$u_3(x_3,r,t) = -\frac{8\nu\omega_E^{-1}\varepsilon\varepsilon_1}{\mu}\left(\frac{r-\varepsilon_1}{\varepsilon}-\frac{(r-\varepsilon_1)^2}{2\varepsilon^2}\right)Q(x_3,t),$$
$$u_r(x_3,r,t) = -\frac{2\omega_E^{-1}\varepsilon\nu}{\lambda+2\mu}\left(\frac{r-\varepsilon_1}{\varepsilon}-\frac{(r-\varepsilon_1)^2}{2\varepsilon^2}\right)\varepsilon_1^2Q(x_3,t).$$

MODIFIED VARIATIONAL FORMULATION FOR NUMERICAL SETUP

Substituting (M) into the following integral identity

$$\omega_\rho \frac{\mathrm{d}}{\mathrm{dt}} \int\limits_{D_\varepsilon^e} r\rho_e \Big(\frac{\partial u_3}{\partial t}\varphi_3 + \frac{\partial u_r}{\partial t}\varphi_r\Big) + \omega_E \int\limits_{D_\varepsilon^e} \Big(2\mu\, r\Big(\frac{\partial u_3}{\partial x_3}\frac{\partial \varphi_3}{\partial x_3}$$

$$+\frac{\partial u_r}{\partial r}\frac{\partial \varphi_r}{\partial r}\Big) + \mu\, r\Big(\frac{\partial u_3}{\partial r} + \frac{\partial u_r}{\partial x_3}\Big)\Big(\frac{\partial \varphi_3}{\partial r} + \frac{\partial \varphi_r}{\partial x_3}\Big) + 2\mu\, r\frac{u_r}{r}\frac{\varphi_r}{r}$$

$$+\lambda r\Big(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r}\Big)\Big(\frac{\partial \varphi_3}{\partial x_3} + \frac{\partial \varphi_r}{\partial r} + \frac{\varphi_r}{r}\Big)\Big)$$

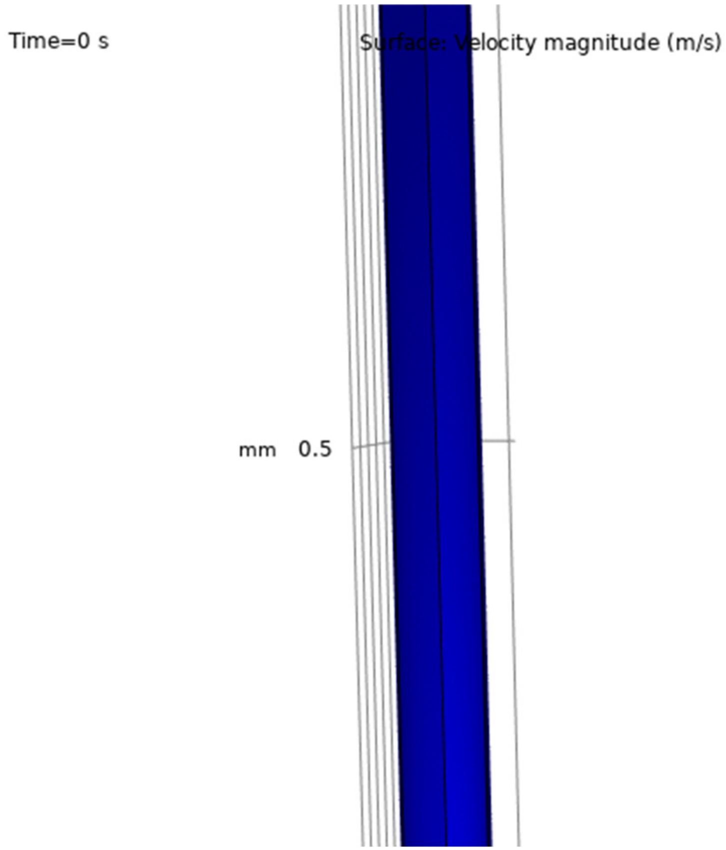
$$+\rho_f \frac{\mathrm{d}}{\mathrm{dt}} \int\limits_{D^f} r\,(v_3\psi_3 + v_r\psi_r) + 2\nu \int\limits_{D^f} r\Big(\frac{\partial v_3}{\partial x_3}\frac{\partial \psi_3}{\partial x_3}$$

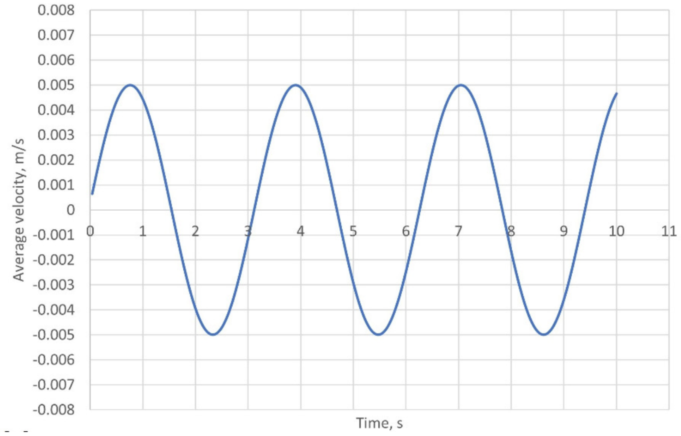
$$+\frac{1}{2}\Big(\frac{\partial v_3}{\partial r} + \frac{\partial v_r}{\partial x_3}\Big)\Big(\frac{\partial \psi_3}{\partial r} + \frac{\partial \psi_r}{\partial x_3}\Big) + \frac{\partial v_r}{\partial r}\frac{\partial \psi_r}{\partial r}\Big) = 0,$$

$$\longrightarrow \quad C_1\frac{\partial^2 Q(x_3,t)}{\partial t^2} + C_2\frac{\partial^4 Q(x_3,t)}{\partial x_3^2\partial t^2} + C_3\frac{\partial Q(x_3,t)}{\partial t} + C_4\frac{\partial^3 Q(x_3,t)}{\partial x_3^2\partial t}$$

$$+C_5\frac{\partial^4 Q(x_3,t)}{\partial x_3^4} + C_6\frac{\partial^2 Q(x_3,t)}{\partial x_3^2} + C_7 Q(x_3,t) = 0,$$

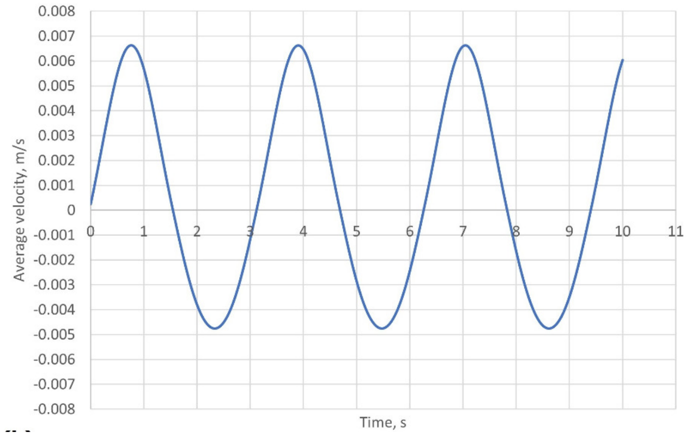
PIPE



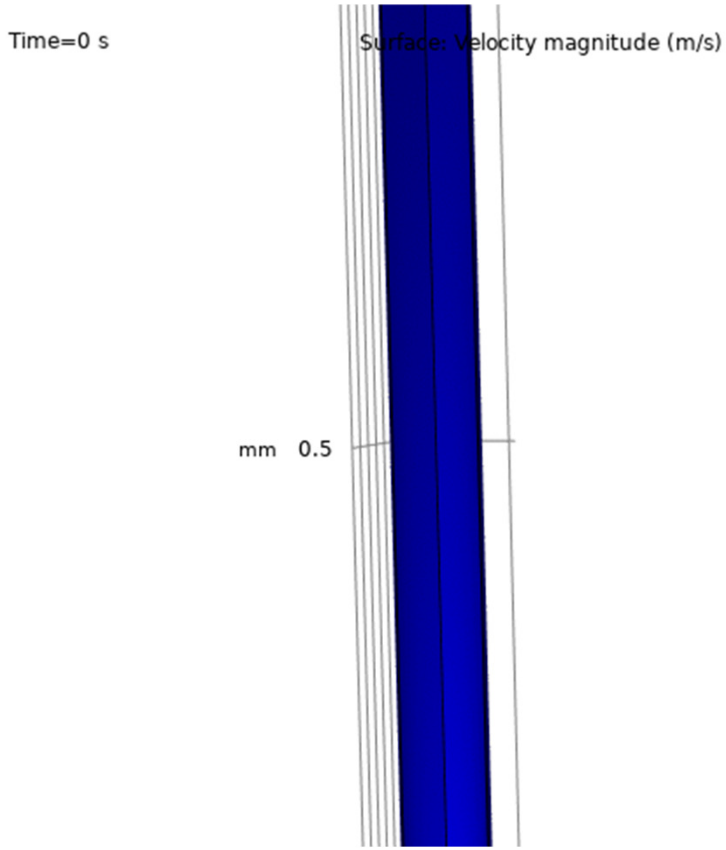


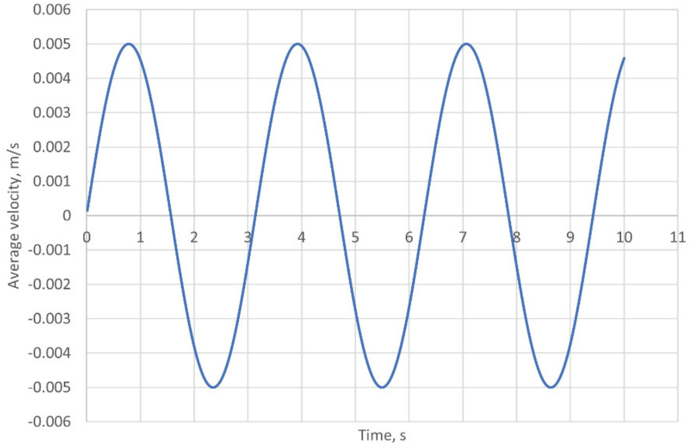
g(inlet)=sin(2t)

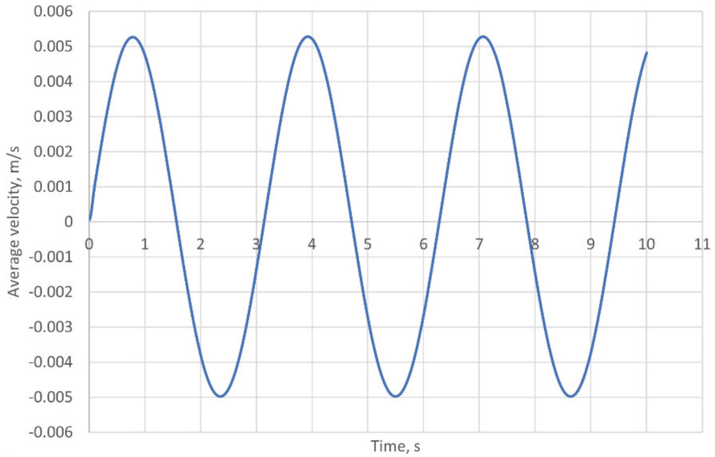
g(outlet)=sin(2t+0.1)



PIPE







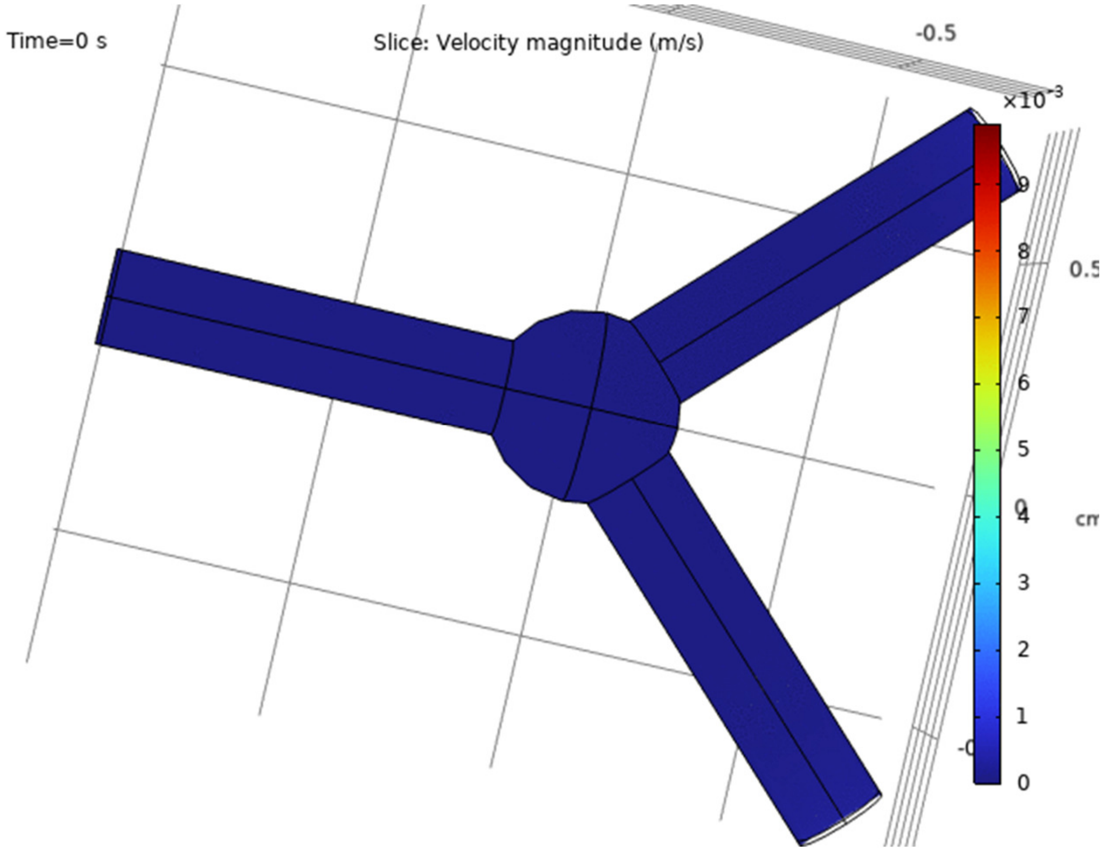
g(inlet)=sin(2t)

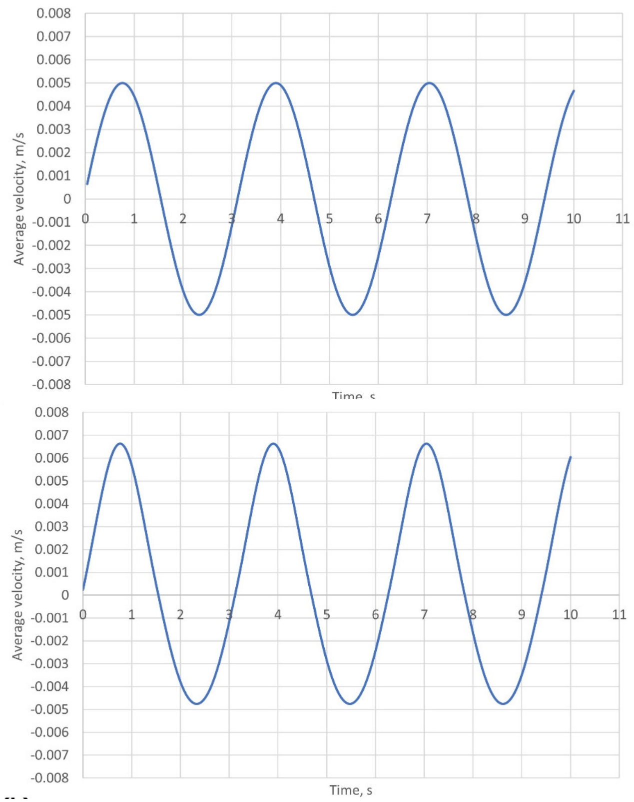
g(outlet)=sin(2t+0.01)

Y-SHAPED NETWORK OF VESSELS

g(inlet)=sin(2t)

g(outlet)=sin(2t+0.1)





Y-SHAPED NETWORK OF VESSELS

g(inlet)=sin(2t)

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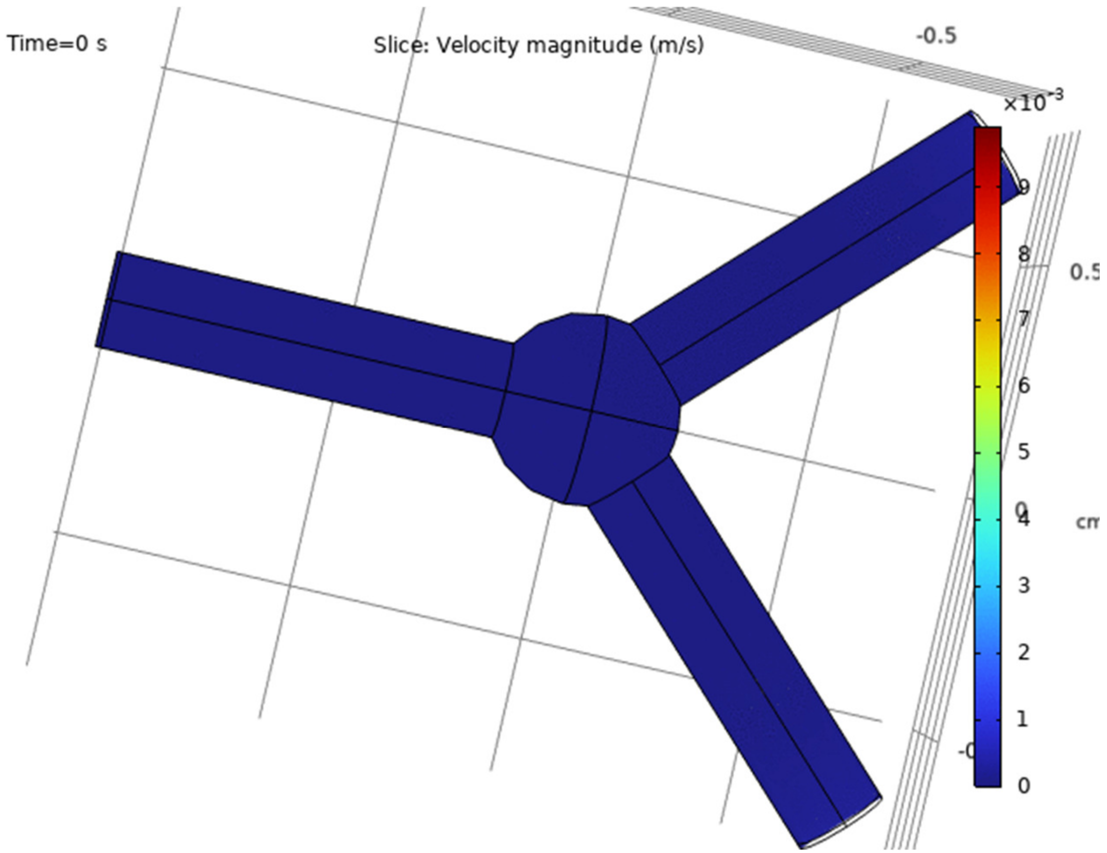
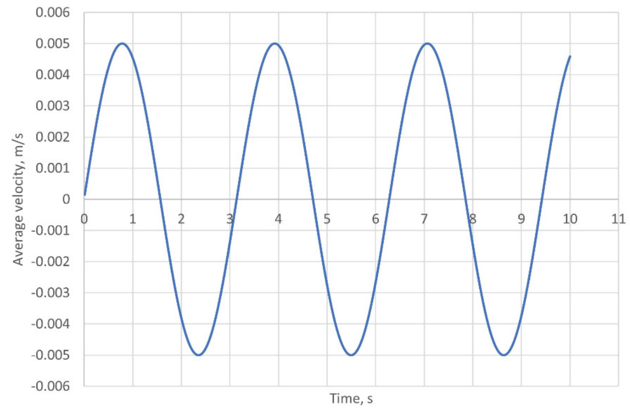
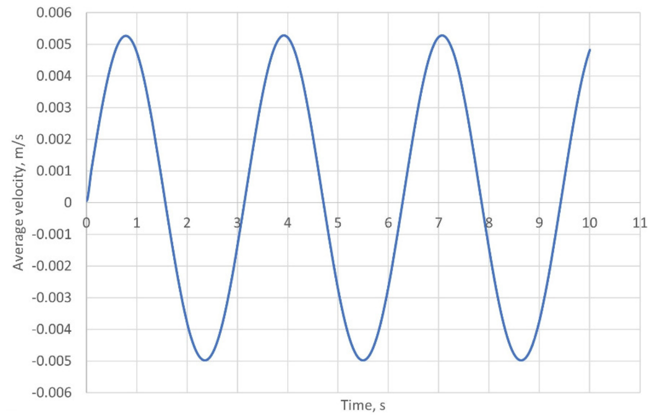
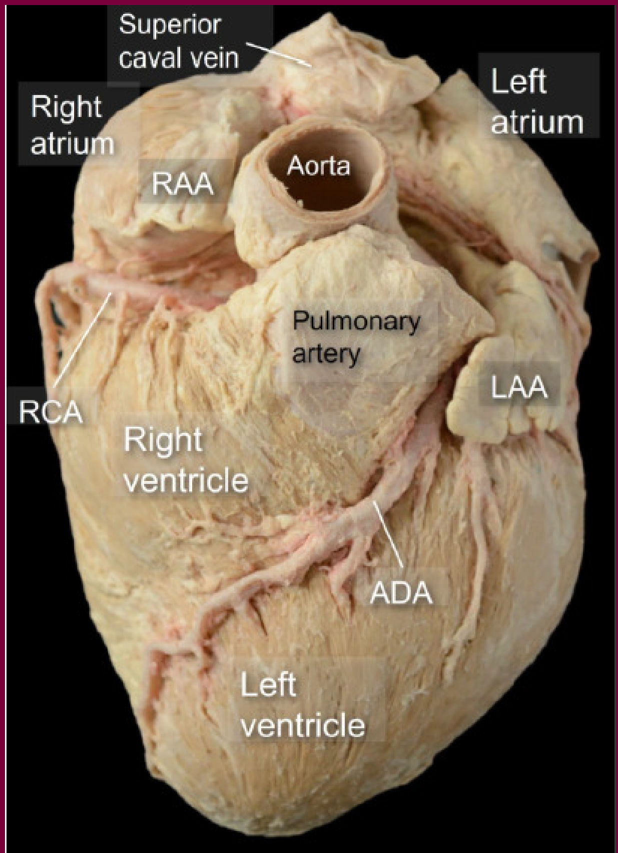






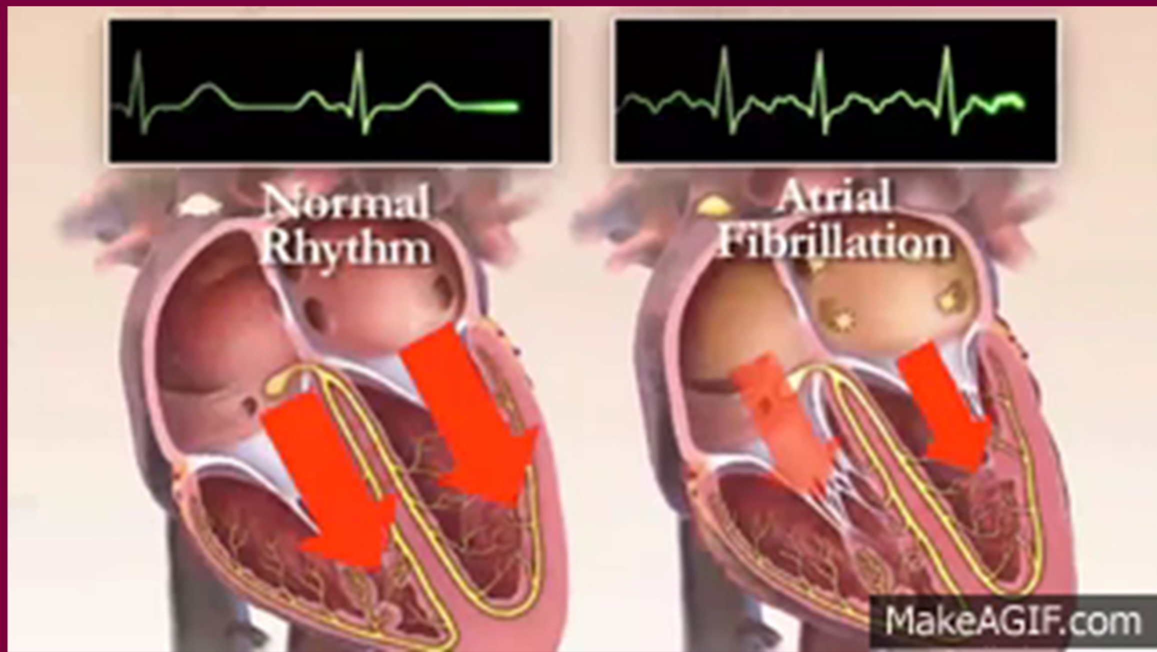
Efficient computation of blood velocity in the left atrial appendage: A practical perspective

MOTIVATION

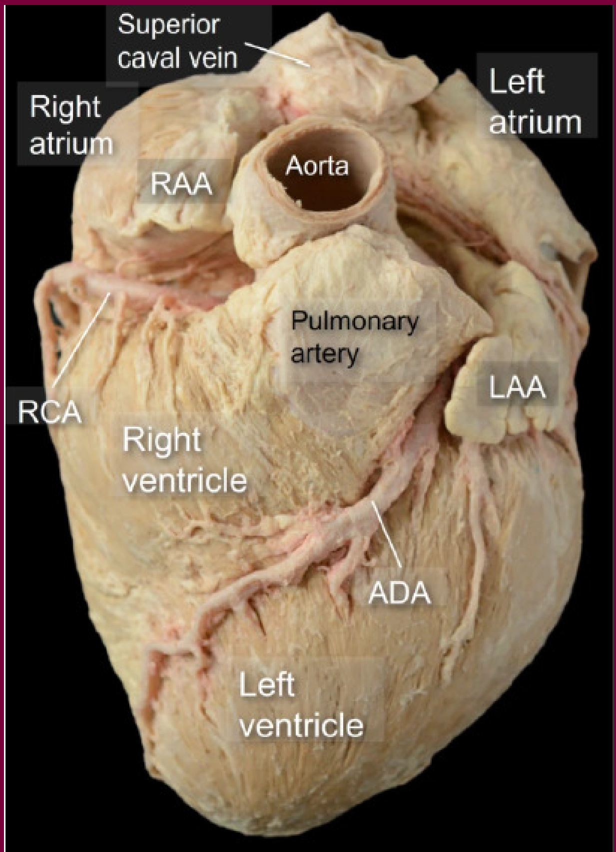


HUMAN HEART

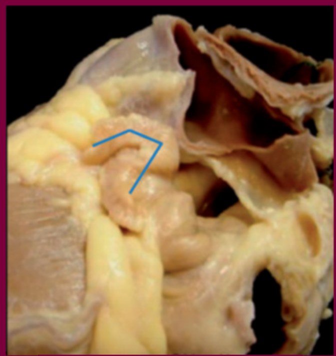
Atrial fibrillation



MOTIVATION

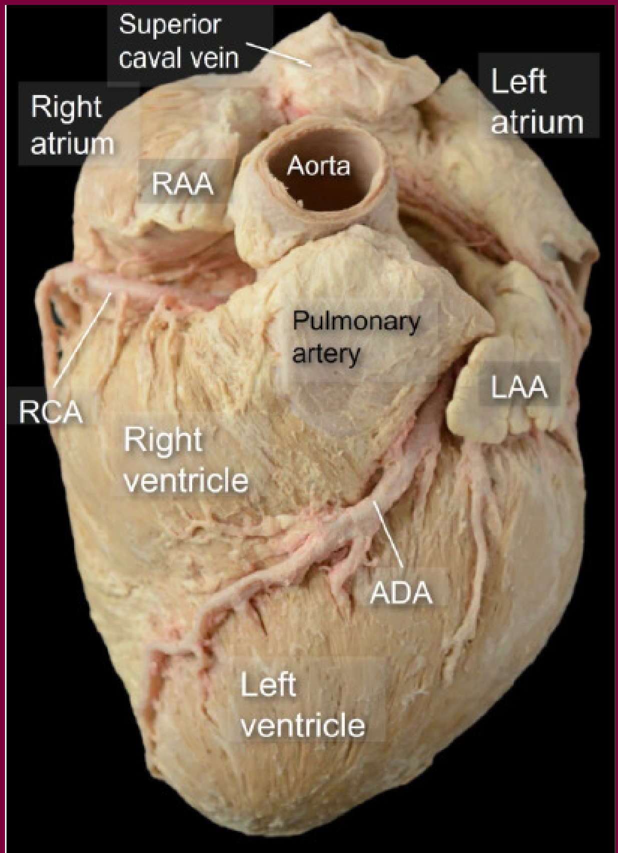


HUMAN
HEART

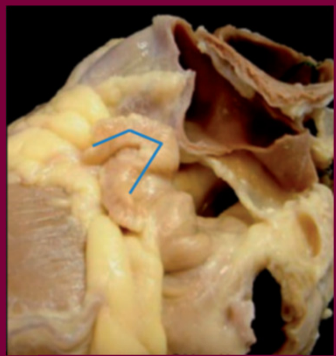


CHICKEN
WING

MOTIVATION



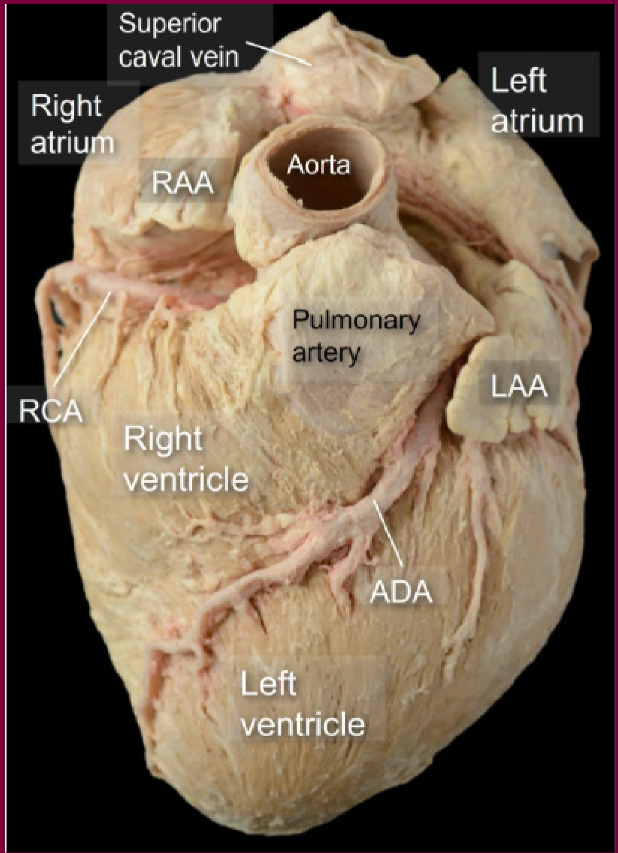
HUMAN HEART



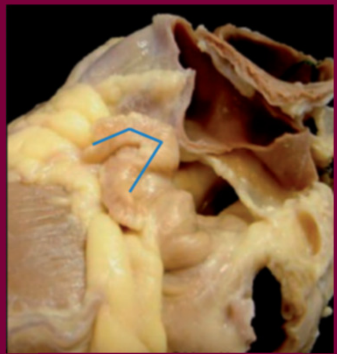
CHICKEN WING

WINDSOCK

MOTIVATION

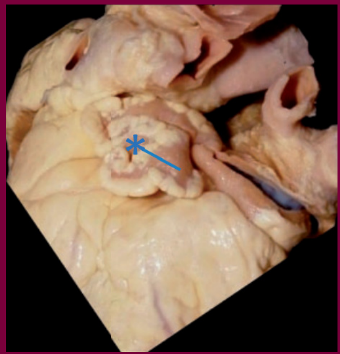


HUMAN HEART



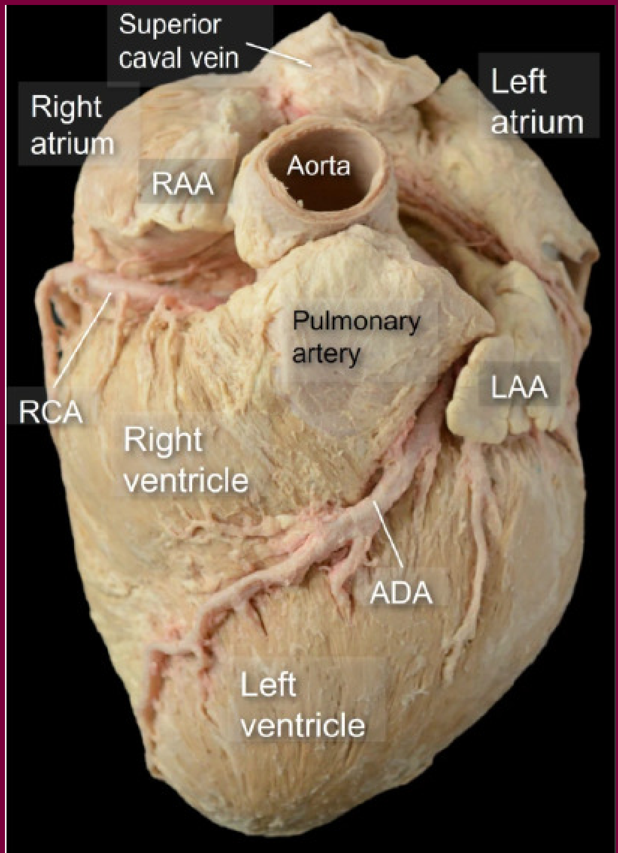
CHICKEN WING

WINDSOCK

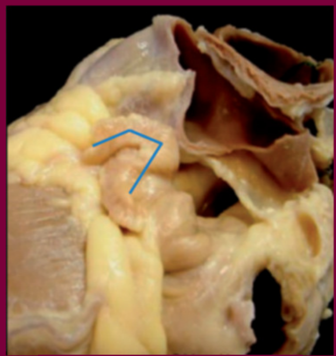


CAULIFLOWER

MOTIVATION

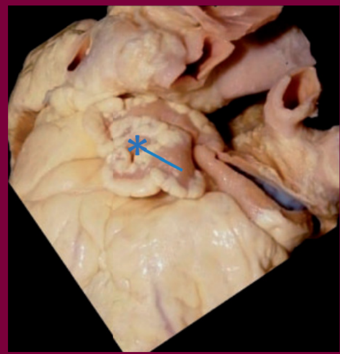


HUMAN HEART

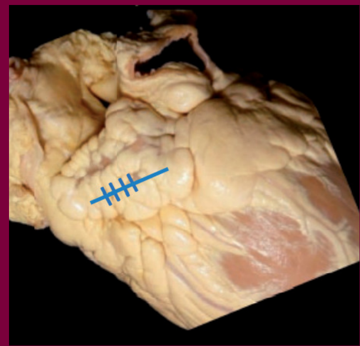


CHICKEN WING

WINDSOCK

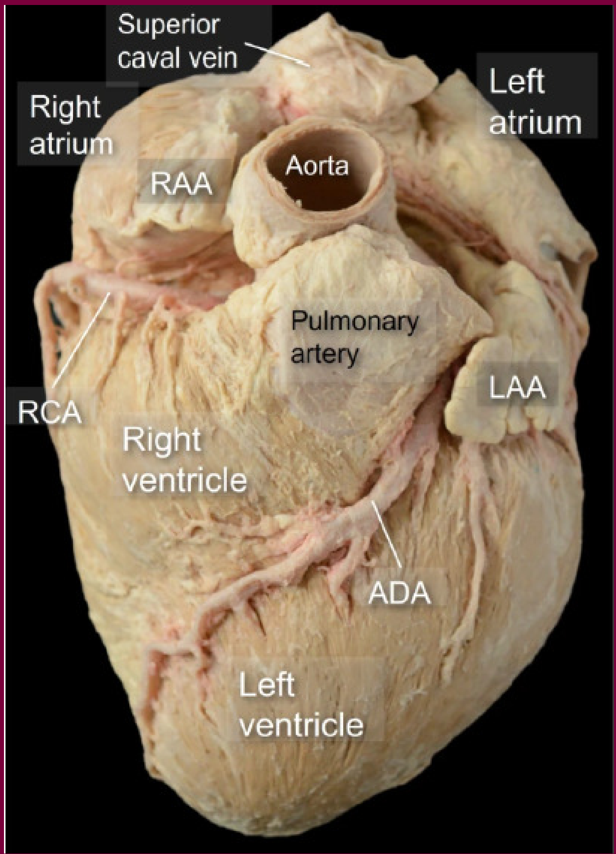


CAULIFLOWER

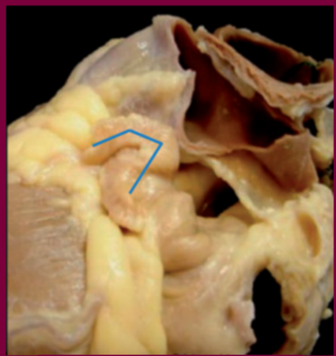


CACTUS

MOTIVATION

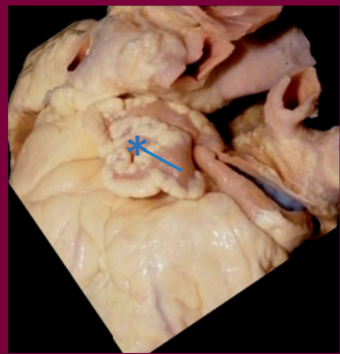


HUMAN HEART

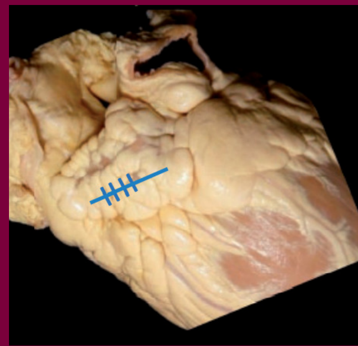


CHICKEN WING

WINDSOCK



CAULIFLOWER



CACTUS

Which geometry leads to the highest chance of thrombi formation and STROKE?

N. Karim *et al.*, The left atrial appendage in humans: structure, physiology, and pathogenesis

CHA_2DS_2–VASc Score		
C	Congestive Heart Failure	1 point
H	Hypertension	1 point
A_2	Age $\geqslant 75$ years	2 points
D	Diabetes	1 point
S_2	Stroke	2 points
V	Vascular disease	1 point
A	Age $\geqslant 65$ years	1 point
Sc	Sex category, female	1 point

CHA_2DS_2–VASc (or $CHADS_2$) score system. Maximum total score = 10 points. ESC 2010 Anticoagulation Recommendations: Score = 0 no therapy or aspirin. Score = 1 aspirin or oral anticoagulation (oral anticioagulation preferred). Score $\geqslant 2$ oral anticoagulation.

IMAGING. CLEANING. GEOMETRY CREATION

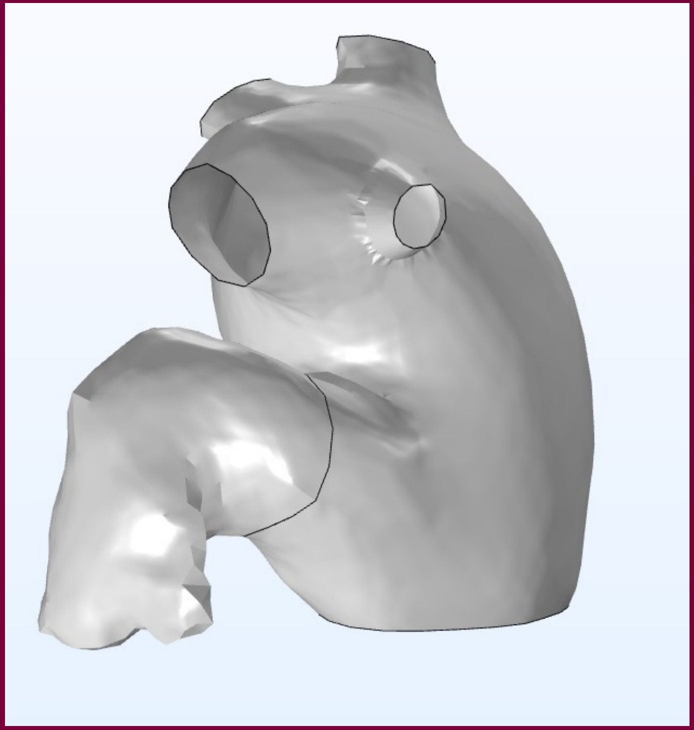
METHODOLOGY:

IMAGING
(CT)

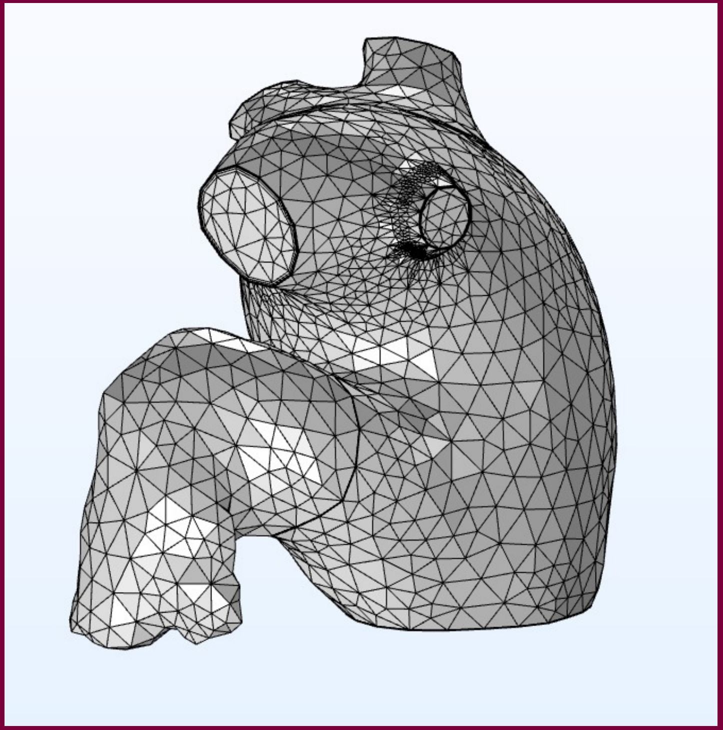
SEGMENTATION (L.VENTRICLE & L.ATRIUM -> ATRIUM) → MESHING

CACTUS

GEOMETRY

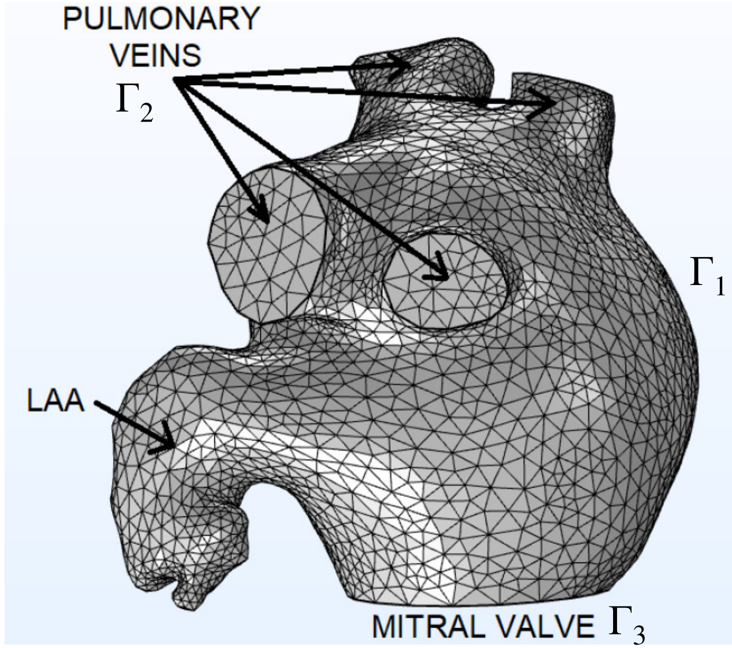


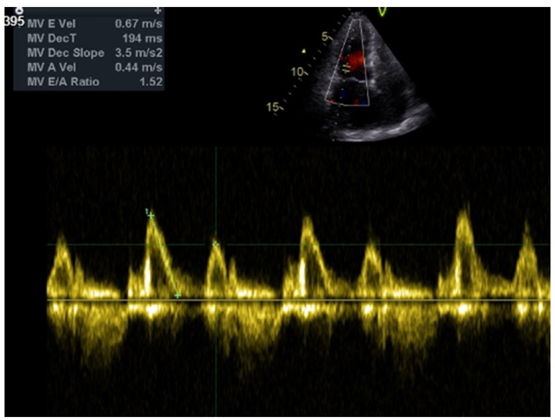
MESH

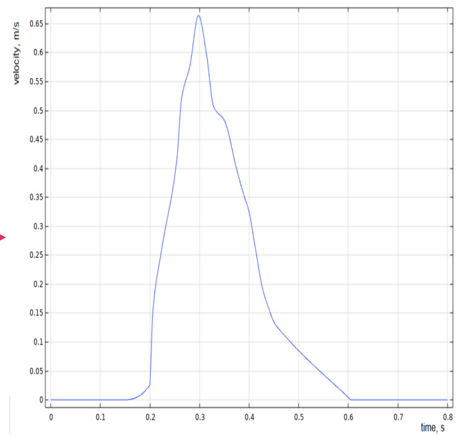


FIRST STEP

$$
\begin{cases}
\rho \mathbf{u}_t - \mu \Delta \mathbf{u} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = 0, \\
\operatorname{div}\ \mathbf{u} = 0, \\
\mathbf{u}|_{\Gamma_1} = 0, \\
\mathbf{u}|_{\Gamma_2} = \mathbf{g}(x,t), \\
\mathbf{u}|_{\tau}|_{\Gamma_3} = 0, \quad p|_{\Gamma_3} = 0, \\
\mathbf{u}(x,0) = 0,
\end{cases}
$$





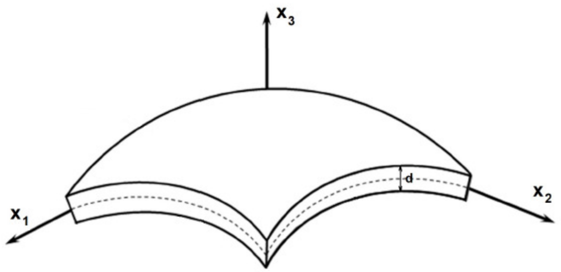


SECOND STEP

In the second step we make computations in a fully coupled model where for a fluid flow we utilize the reference velocity obtained in the first step and the equation of motion from shell theory. The FSI code applies the Uflyand-Mindlin shell theory for the elastic wall (in our case myocardium). Namely, the displacement vector $\mathbf{u}$ is expressed in the local coordinates in the following way:

$$\mathbf{u}(x_1,x_2,x_3,t) = \boldsymbol{\eta}(x_1,x_2,t) + x_3\zeta(x_1,x_2,t),$$

where x_1 and x_2 are coordinates in the plane of the shell, x_3 is a normal coordinate , $\boldsymbol{\eta}(x_1,x_2,t)$ is the displacement vector of the shell and $\zeta(x_1,x_2,t)$ is the displacement of shell normal.



The equation of motion where the divergence of stress equals the volume force is as follows:

$$\rho\Big(\frac{\partial^2\boldsymbol{\eta}}{\partial t^2} + z\frac{\partial^2\zeta}{\partial t^2}\Big) = \nabla\cdot(J\sigma\boldsymbol{F}^{-T})^T + \mathbf{F}_V + 6(\mathbf{M}_V\times\mathbf{n})\frac{z}{d},$$

where $\mathbf{F}_V = \frac{\mathbf{F}_A}{d}$; $\mathbf{M}_V = \frac{\mathbf{M}_A}{d}$, $z = \frac{2x_3}{d}$; $\mathbf{F}$ is the deformation gradient; $J\sigma\boldsymbol{F}^{-T}$ is the 1^{st} Piola-Kirchhoff stress, $J = \det\boldsymbol{F}$ is the Jacobian determinant; d is the thickness of the wall, ρ is density of the wall, $\mathbf{M}_A$ – moment, $\mathbf{K}$ — viscous stress tensor. The local z coordinate $[-1,1]$ for thickness dependent results z. Its value can be changed from -1 (downside) to $+1$ (upside). A value of 0 means the midsurface of the shell. This is the default position for stress and strain evaluation during the analysis of the results. Moreover if we use a cross product rule for moment we obtain:

$$\mathbf{M}_A\times\mathbf{n} = [M_{22} \quad -M_{11} \quad 0]^T,$$

where $M_{ij} = \displaystyle\int_{-d/2}^{d/2} x_3\sigma_{ij}dx_3$ and $\mathbf{n} = [0 \quad 0 \quad 1]^T$.

SECOND STEP

The junction conditions equating the normal stresses and the velocity at the boundary of the reference configuration (i.e. when x belongs the interface):

$$\mathbf{F}_A = \left(-p_{\text{wall}}\mathbf{I} - [-p\mathbf{I} + \mathbf{K}]\right) \cdot \mathbf{n},$$

and the velocity of a moving wall (translational velocity) is

$$\mathbf{u}(x + \boldsymbol{\eta}(x,t), t) = \frac{\partial \boldsymbol{\eta}}{\partial t}.$$

We take into account, that the average stress tensor of the unloaded shell $\langle \sigma_z \rangle = \int\limits_{-1}^{1} \sigma_z dz = \frac{2}{d} \int\limits_{-d/2}^{d/2} \sigma_{x_3} dx_3 = 0.$

Since the strain tensor (see [56]):

$$\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)$$

and stress tensor

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij},$$

where λ and μ are Lamé parameters, δ_{ij} is Kronecker coefficient:

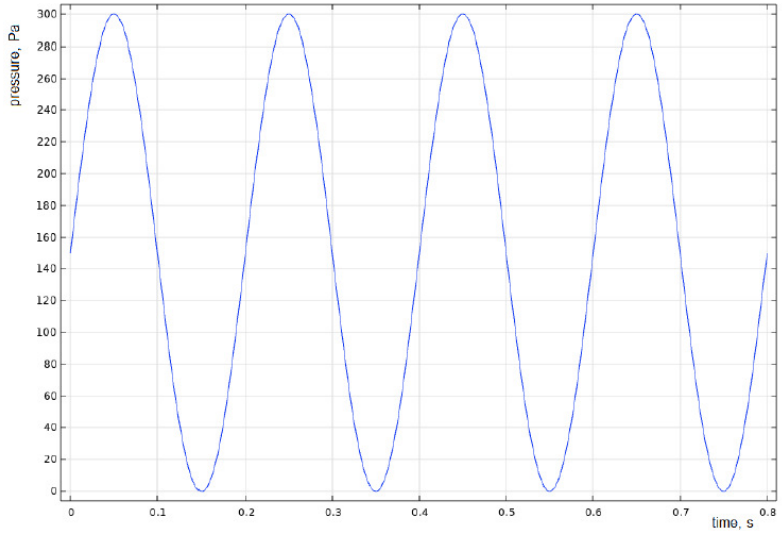
$$\delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j, \end{cases}$$

$$\sigma_{31} = \mu\Big(\frac{\partial \eta_3}{\partial x_1} + x_3 \frac{\partial \zeta_3}{\partial x_1} + \zeta_1\Big),$$

$$\sigma_{32} = \mu\Big(\frac{\partial \eta_3}{\partial x_2} + x_3 \frac{\partial \zeta_3}{\partial x_2} + \zeta_2\Big),$$

$$\sigma_{33} = 2\mu\zeta_3 + \lambda\Big(\frac{\partial \eta_1}{\partial x_1} + x_3 \frac{\partial \zeta_1}{\partial x_1} + \frac{\partial \eta_2}{\partial x_2} + x_3 \frac{\partial \zeta_2}{\partial x_2} + \zeta_3\Big).$$

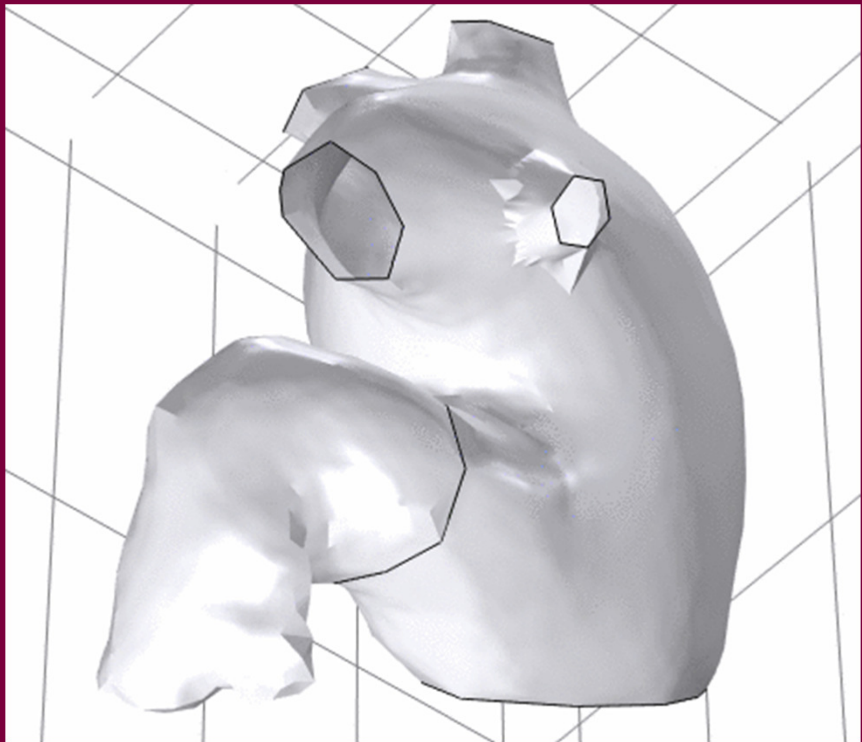
We prescribe the total pressure on the surface of the shell

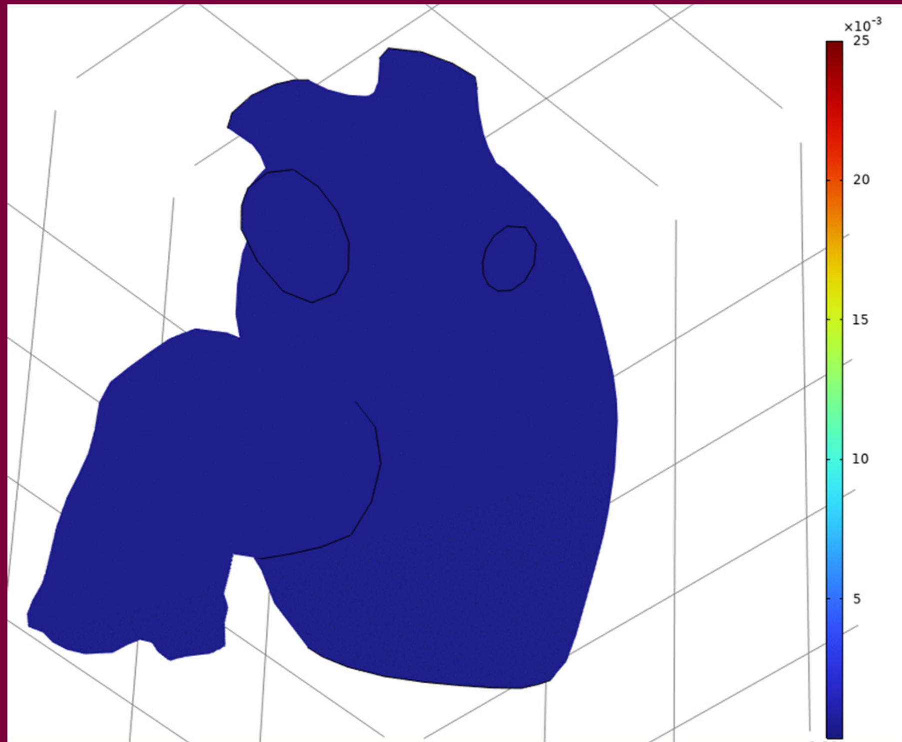


SIMULATIONS

PATIENT-SPECIFIC COMPUTER FSI SIMULATION FOR CACTUS LEFT ATRIUM GEOMETRY

SINUS RHYTHM

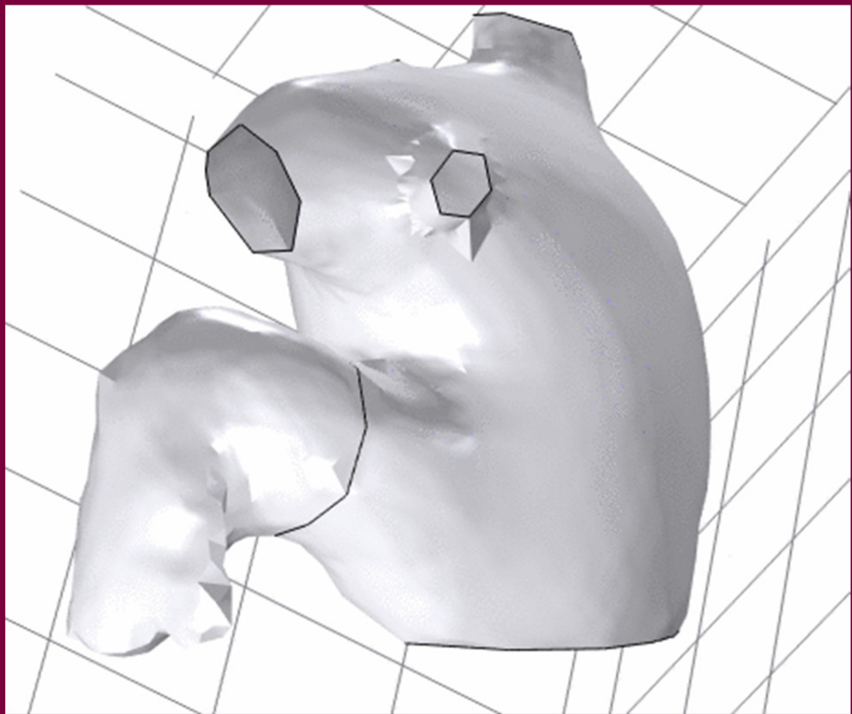


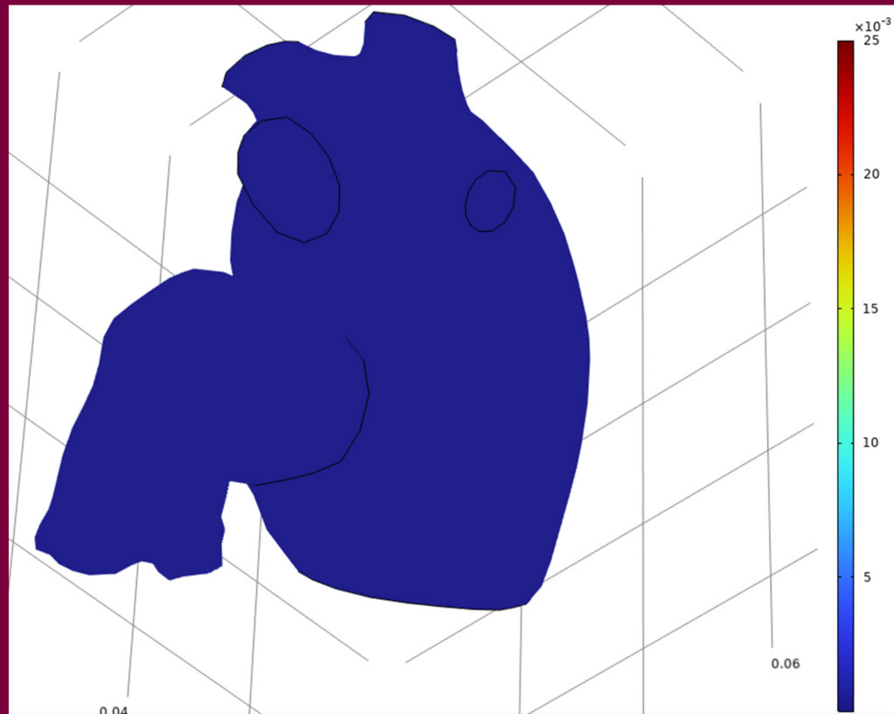


SIMULATIONS

PATIENT-SPECIFIC COMPUTER FSI SIMULATION FOR CACTUS LEFT ATRIUM GEOMETRY

ATRIAL FIBRILLATION

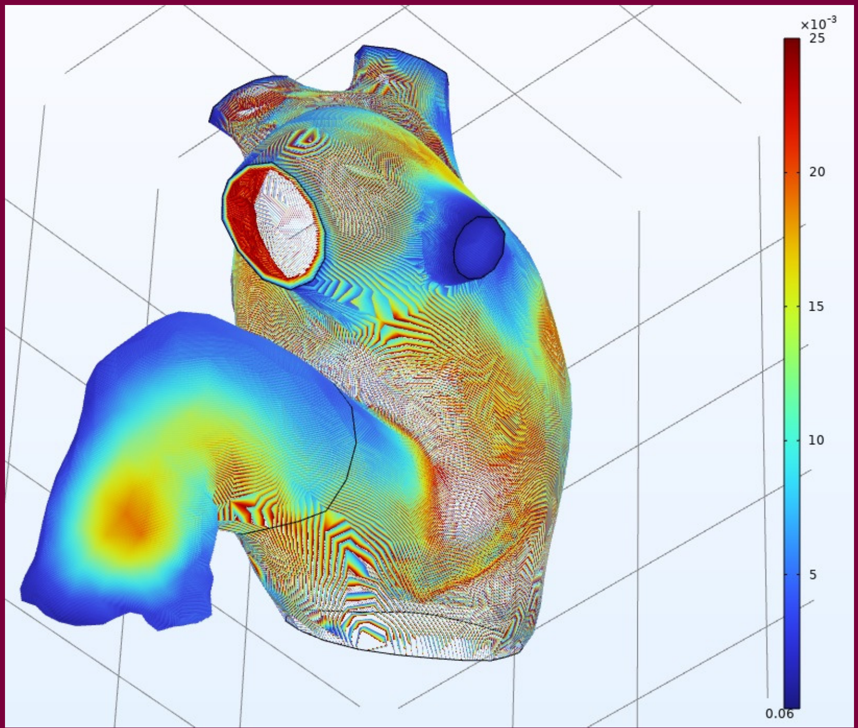




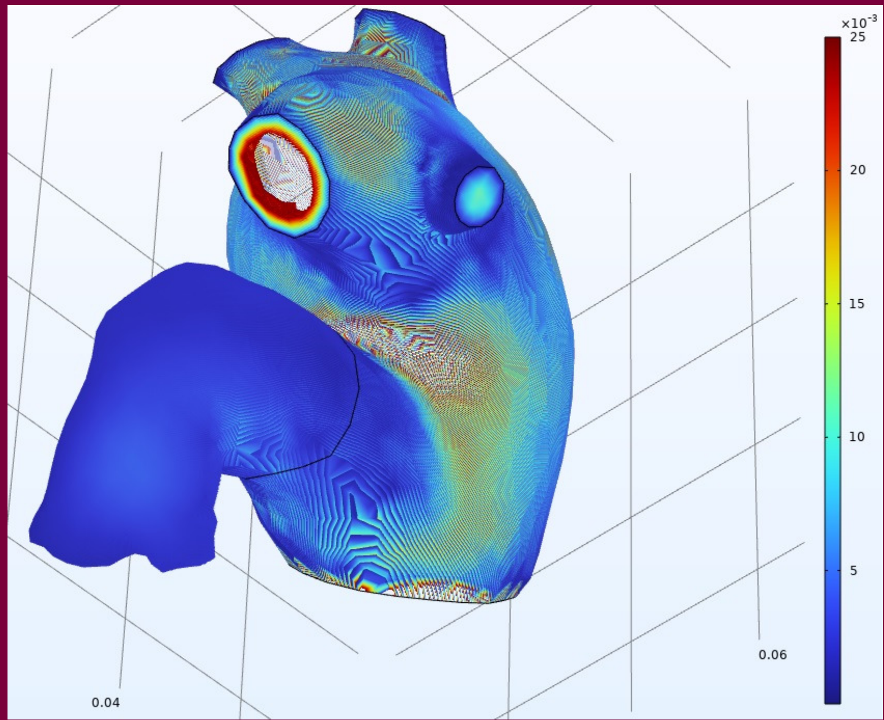
SIMULATIONS

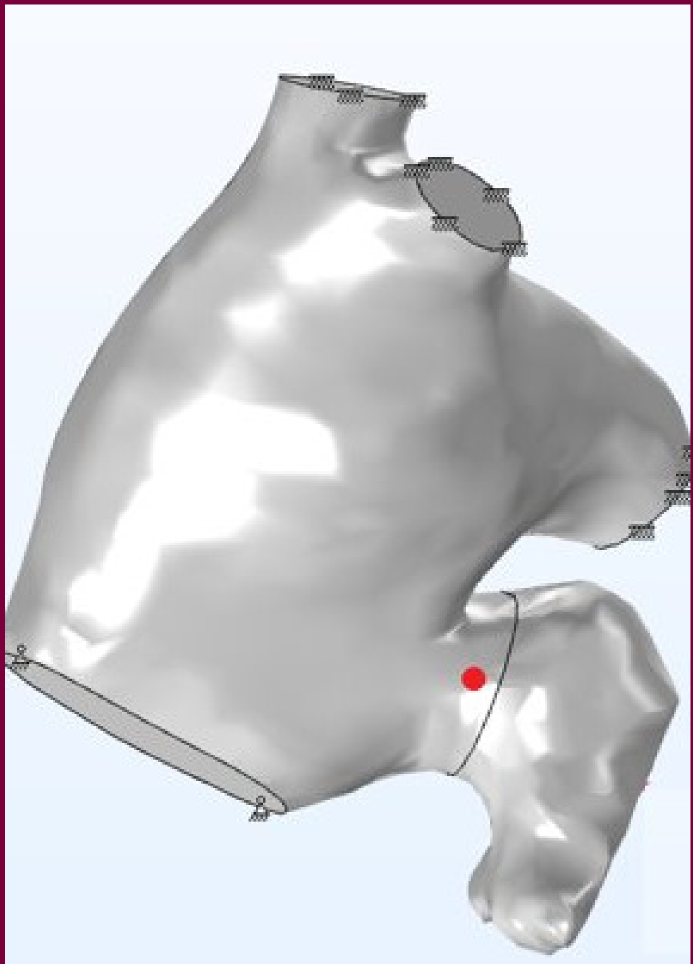
PATIENT-SPECIFIC COMPUTER FSI SIMULATION FOR CACTUS LEFT ATRIUM GEOMETRY

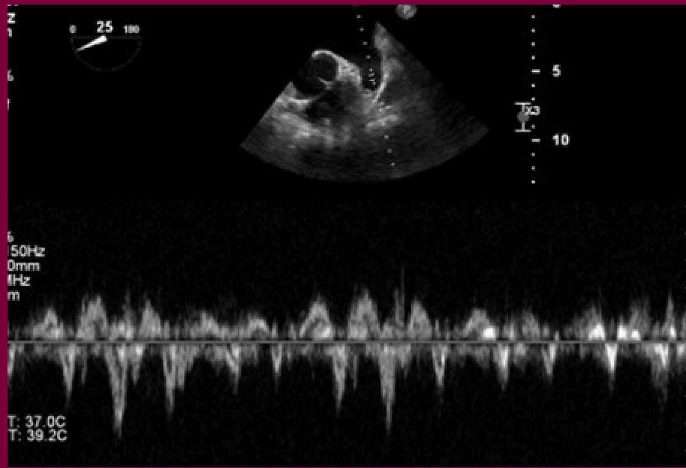
SINUS RHYTHM

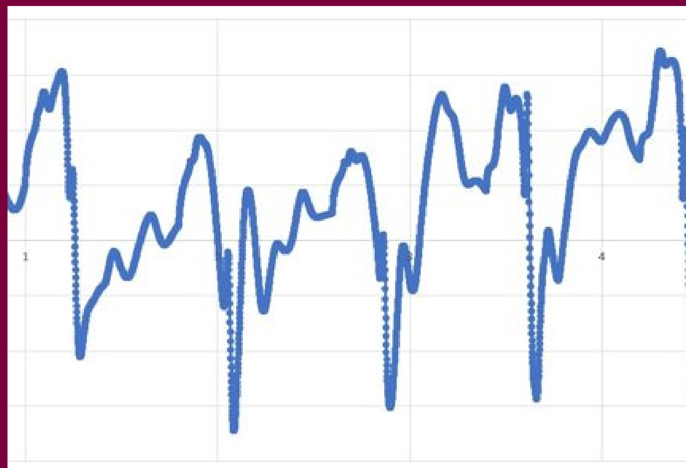


ATRIAL FIBRILLATION







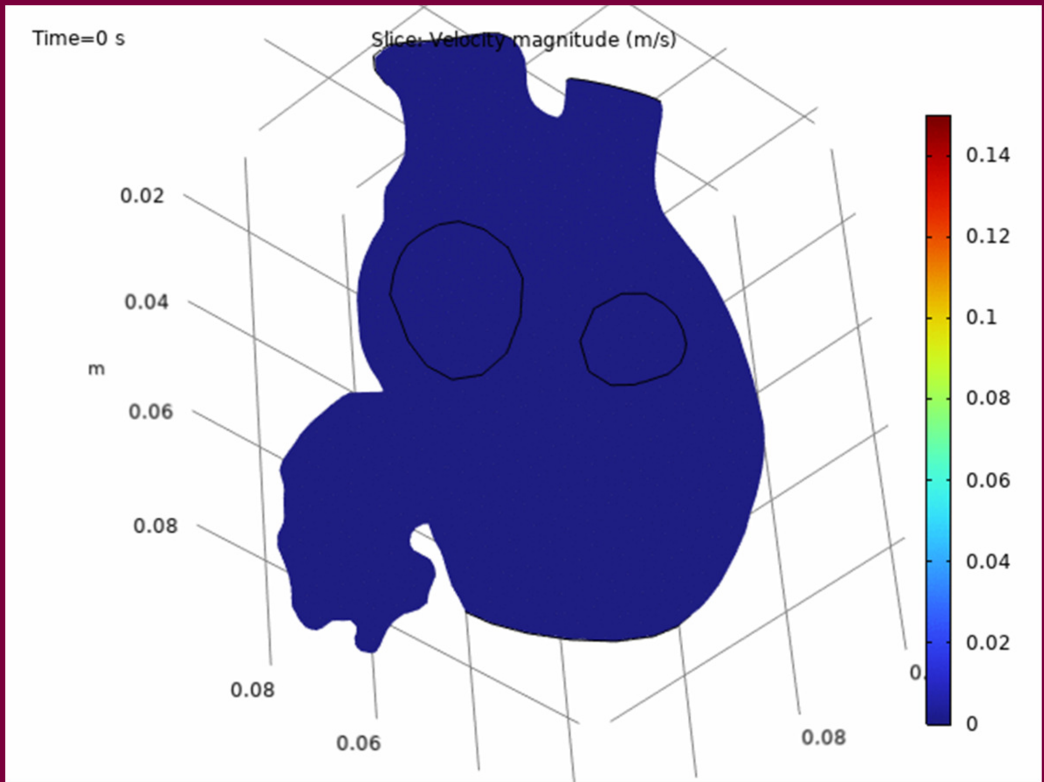


ATRIAL FIBRILLATION INLET OF LAA

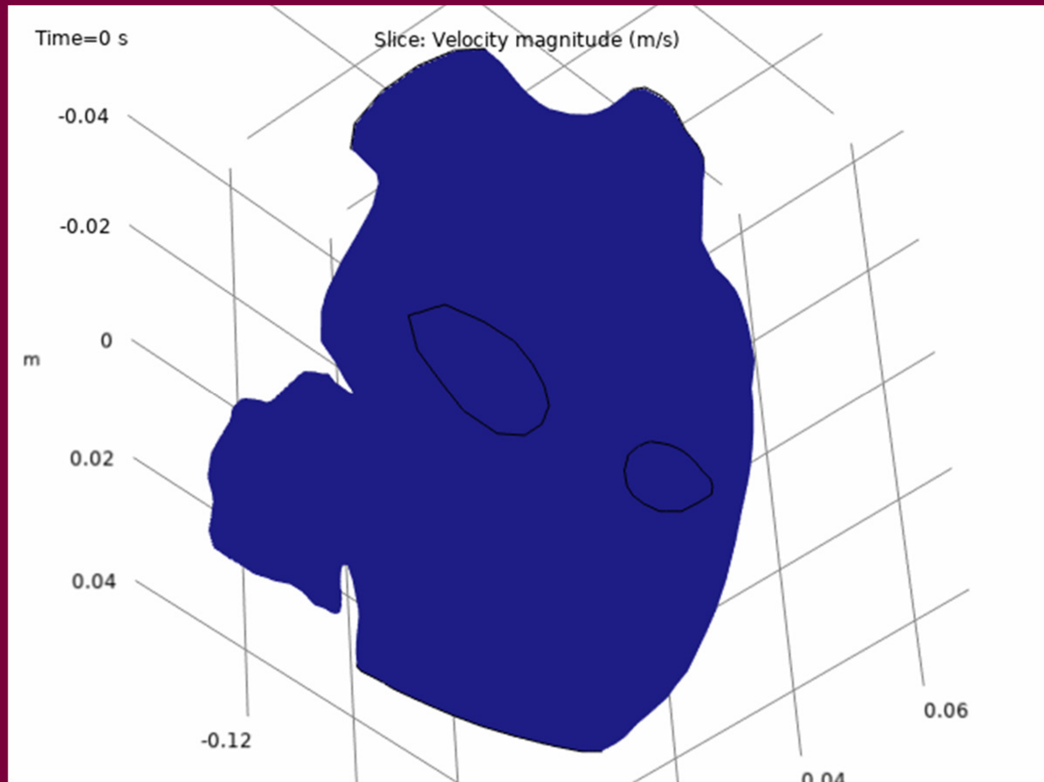
SIMULATIONS

PATIENT-SPECIFIC COMPUTATION OF BLOOD FLOW VELOCITY IN THE LA

STROKE



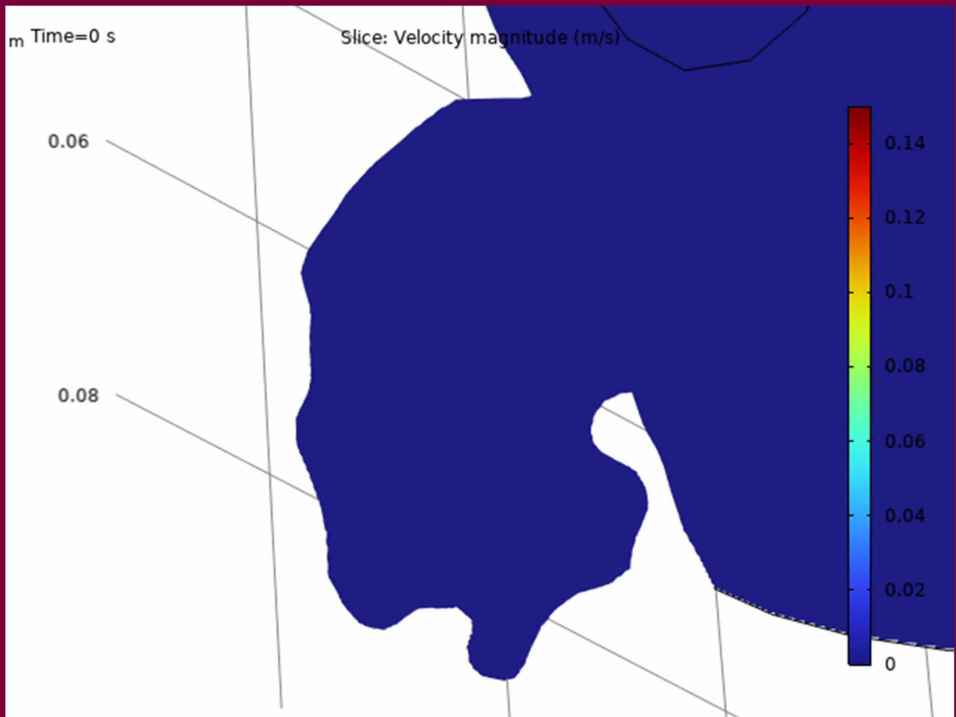
NO STROKE



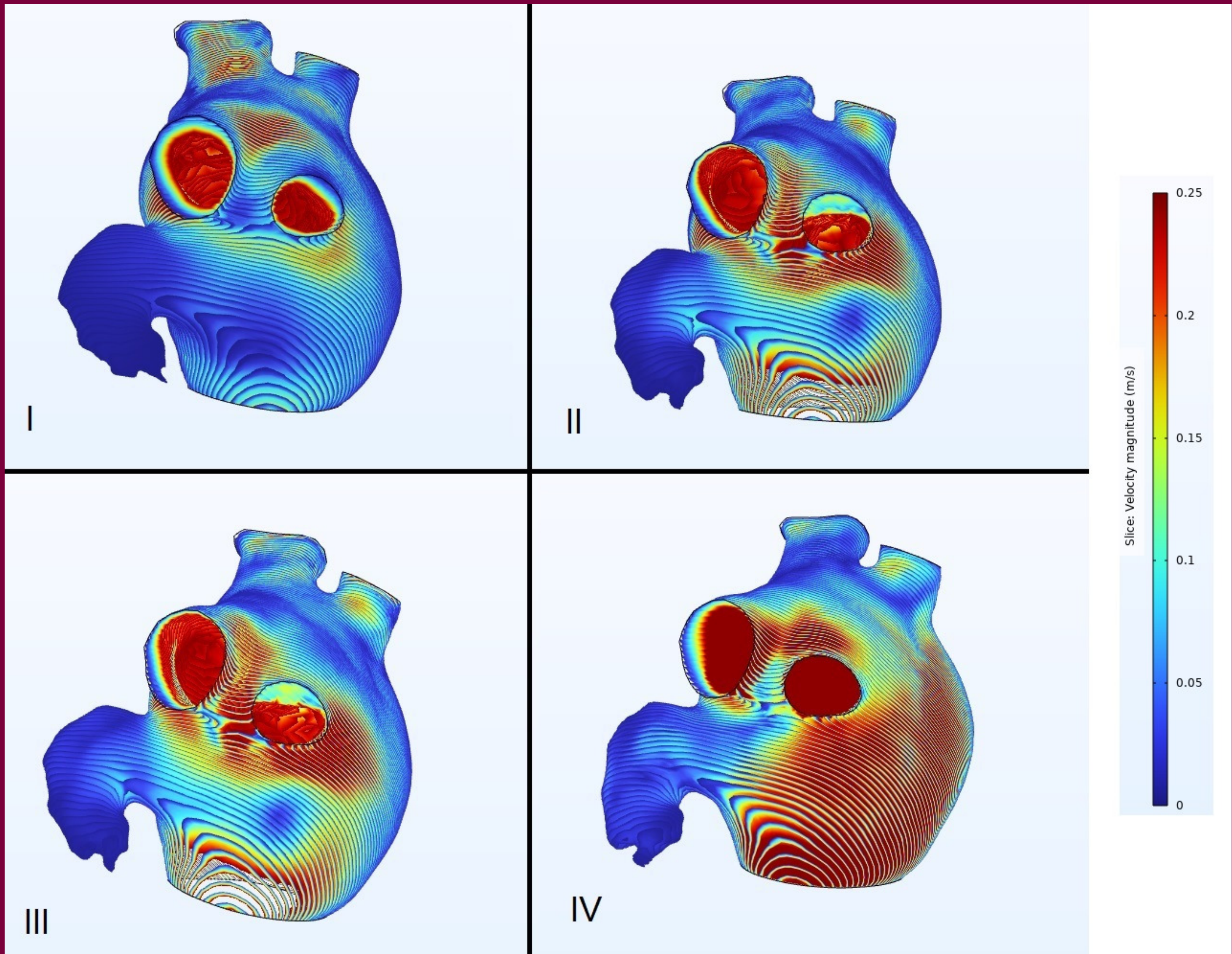
SIMULATIONS

PATIENT-SPECIFIC COMPUTATION OF BLOOD FLOW VELOCITY IN THE LAA

STROKE



NO STROKE



Blood velocity magnitude in LA, when the angle between LA and LAA, I – 30°, II – 50°, III – 70°, IV – 90°