ADI scheme for partially dimension reduced heat conduction models

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Mathematical Model in 3D

Let us assume that initial and boundary conditions and all coefficients satisfy the radial symmetry condition, thus we get the following problem

$$\frac{\partial u}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2}+f(r,z,t),\ (r,z,t)\in Q_T=\Omega\times(0,T], \quad (1)$$

$$u(r,0,t)=g_1(r,t),\ u(r,l,t)=g_2(r,t),\ (r,t)\in(0,R]\times(0,T], \quad (2)$$

$$r\frac{\partial u}{\partial r}=0, \quad 0<z<l, r=0 \text{ and } r=R,\ 0<t\leq T, \quad (3)$$

$$u(r,z,0)=u^0(r,z), \quad (r,z)\in\Omega. \quad (4)$$

Let $S(u)$

$$S(u) = \frac{2}{R^2} \int_0^R ru(r,z,t)dr$$

denote the averaging operator.

We assume that the initial condition u^0 and source function f satisfy the relations

$$u^0(r,z) = S(u^0), \quad f(r,z,t) = S(f), \quad (z,t) \in (0,l) \times (0,T].$$

It means that u^0 and f do not depend on r within the tube $\mathcal{T}$

Denote a reduced tube $\mathcal{T}_\delta = D \times (\delta, l - \delta)$ and $\Omega_\delta = \{(r,z) \in (0,R) \times (\delta, l - \delta)\}$.

Function U is called an approximate solution to problem (1) – (4) if it satisfies the following problem

$$\frac{\partial U}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{\partial^2 U}{\partial z^2}+f(z,t),\ (r,z,t)\in(\Omega\setminus\Omega_\delta)\times(0,T], \quad (5)$$

$$\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial z^2}+f(z,t),\quad (r,z,t)\in\Omega_\delta\times(0,T], \quad (6)$$

$$U(r,0,t)=g_1(r,t),\ U(r,l,t)=g_2(r,t),\ (r,t)\in(0,R]\times(0,T], \quad (7)$$

$$r\frac{\partial U}{\partial r}=0,\ z\in(0,\delta)\cup(l-\delta,l),\ r=0, r=R,\ 0<t\leq T, \quad (8)$$

$$U(r,z,0)=u^0(r,z),\quad (r,z)\in\Omega. \quad (9)$$

In $\Omega_\delta \times (0, T]$ the solution U do not depend on r, i.e.

$$U(r,z,t)=S(U),\quad (r,z,t)\in\Omega_\delta\times(0,T].$$

From the weak form of the heat equation, it follows that the conjugation conditions are valid at the truncations of the tube

$$U\Big|_{z=\delta-0} = U\Big|_{z=\delta+0}, \quad U\Big|_{z=l-\delta-0} = U\Big|_{z=l-\delta+0}, \tag{10}$$

$$\frac{\partial S(U)}{\partial z}\Big|_{z=\delta-0} = \frac{\partial U}{\partial z}\Big|_{z=\delta+0}, \ \frac{\partial U}{\partial z}\Big|_{z=l-\delta-0} = \frac{\partial S(U)}{\partial z}\Big|_{z=l-\delta+0}. \tag{11}$$

The conditions (10) are classical and mean that U is continuous at the truncation points.
The remaining two conditions (11) are nonlocal and they define the conservation of full fluxes along the separation lines.

For functions defined on the grid $\Omega_h \times \omega_t$ we introduce the discrete operators with respect to z and r:

$$\partial_z U^n_{jk} := \frac{U^n_{jk} - U^n_{j,k-1}}{H}, \quad A^h_2 U^n_{jk} := -\frac{1}{H}\Big(\partial_z U^n_{j,k+1} - \partial_z U^n_{j,k}\Big).$$

$$\partial_r U^n_{jk} := \frac{U^n_{jk} - U^n_{j-1,k}}{h}, \; A^h_1 U^n_{jk} := -\frac{1}{\widetilde{r}_j h}\Big(r_{j+\frac{1}{2}} \partial_r U^n_{j+1,k} - r_{j-\frac{1}{2}} \partial_r U^n_{j,k}\Big),$$

where

$$\widetilde{r}_0 = \frac{1}{8}h, \; \widetilde{r}_j = r_j, \; 1 \le j < J, \; \widetilde{r}_J = \frac{1}{2}\Big(R - \frac{h}{4}\Big), \; r_{-\frac{1}{2}} = 0, \; r_{J+\frac{1}{2}} = 0.$$

Then the heat conduction problem (1)-(4) is approximated by the following Alternating Direction Implicit (ADI) scheme

$$\frac{U_{jk}^{n+\frac{1}{2}} - U_{jk}^{n}}{\tau/2} + A_1^h U_{jk}^{n+\frac{1}{2}} + A_2^h U_{jk}^{n} = f_{jk}^{n+\frac{1}{2}}, \quad (r_j, z_k) \in \bar{\omega}_r \times \omega_z, \quad (12)$$

$$\frac{U_{jk}^{n+1} - U_{jk}^{n+\frac{1}{2}}}{\tau/2} + A_1^h U_{jk}^{n+\frac{1}{2}} + A_2^h U_{jk}^{n+1} = f_{jk}^{n+\frac{1}{2}}, \quad (r_j, z_k) \in \bar{\omega}_r \times \omega_z.$$

LEMMA

If a solution of the problem (1)-(4) *is sufficiently smooth, then the approximation error of ADI scheme* (12) *is* $O(\tau^2 + h^2 + H^2)$.

Lemma

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Proof.

The solution of ADI scheme (12) satisfies the scheme

$$\frac{U_{jk}^{n+1} - U_{jk}^{n}}{\tau} + A_1^h\Big(\frac{U_{jk}^{n+1} + U_{jk}^{n}}{2}\Big) + A_2^h\Big(\frac{U_{jk}^{n+1} + U_{jk}^{n}}{2}\Big) + \frac{\tau^2}{4}A_1^h A_2^h\Big(\frac{U_{jk}^{n+1} - U_{jk}^{n}}{\tau}\Big) = f_{jk}^{n+\frac{1}{2}}.$$

which is equivalent to the classical symmetrical finite difference scheme. □

LEMMA

The discrete operators A_1^h and A_2^h are symmetric and non-negative and positive definite operators, respectively.

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PROOF.

Applying the summation by parts formula and taking into account the boundary conditions for vectors u, v we get

$$(A_2^h u, v) = \sum_{k=1}^{K-1} (A_2^h u)_k v_k H = (\partial_z u, \partial_z v].$$

It follows that A_2^h is symmetric. Since $A_2^h \phi_l = \lambda_l \phi_l$ has a complete set of eigenvectors ϕ_l, $l = 1, \ldots, K-1$, and all eigenvalues are positive $\lambda_l > 0$, A_2^h is a positive-definite operator. □

PROOF.

Now we consider the operator A_1^h. Applying the summation by parts formula we get

$$[A_1^h u, v]_r = (\partial_r u, \partial_r v]_r.$$

It follows from the obtained equality, that A_1^h is symmetric operator.

The eigenvalue problem $A_1^h \psi_l = \mu_l \psi_l$ has a complete set of eigenvectors ψ_l, $l = 0, \dots, J$, one eigenvalue $\mu_0 = 0$ and the remaining eigenvalues are positive $\mu_l > 0$.

□

Lemma

ADI scheme (12) *is unconditionally stable.*

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Proof.

The Fourier stability analysis is used. Let us consider the solution of ADI scheme (12) in the case when boundary conditions $g_j = 0$, $j = 1, 2$. Since operators A_1^h and A_2^h commute, the solution of (12) can be written as

$$U_{jk}^{n+1} = \sum_{l=0}^{J} \sum_{r=0}^{K-1} c_{lr}^{n+1} \psi_l(r_j) \phi_r(z_k).$$

Substituting this formula into equations (12) we obtain the stability equations for each mode

$$c_{lr}^{n+1} = q_{lr} c_{lr}^{n}, \quad q_{lr} = \frac{(1 - 0.5\tau\lambda_r)(1 - 0.5\tau\mu_l)}{(1 + 0.5\tau\lambda_r)(1 + 0.5\tau\mu_l)}.$$

Since eigenvalues $\lambda_r > 0$, $\mu_l \geq 0$, then the ADI scheme (12) is unconditionally stable in the L_2 norm.

The backward Euler scheme

The heat conduction problem (5)-(11) is approximated by the backward Euler scheme

$$\frac{U_{jk}^{n+1}-U_{jk}^{n}}{\tau}+A_1^h U_{jk}^{n+1}+A_2^h U_{jk}^{n+1}=f_{jk}^{n+1},\ (r_j,z_k)\in\bar{\omega}_r\times(\omega_{z1}\cup\omega_{z3}), \tag{13}$$

$$\frac{U_{0k}^{n+1}-U_{0k}^{n}}{\tau}+A_2^h U_{0k}^{n+1}=f_{0k}^{n+1},\quad z_k\in\omega_{z2},$$

$$\frac{U_{0K_1}^{n+1}-U_{0K_1}^{n}}{\tau}+\frac{1}{H^2}\Big(-S_h(U_{K_1-1}^{n+1})+2U_{0K_1}^{n+1}-U_{0,K_1+1}^{n+1}\Big)=f_{0K_1}^{n+1}, \tag{14}$$

$$\frac{U_{0K_2}^{n+1}-U_{0K_2}^{n}}{\tau}+\frac{1}{H^2}\Big(-S_h(U_{K_2+1}^{n+1})+2U_{0K_2}^{n+1}-U_{0,K_2-1}^{n+1}\Big)=f_{0K_2}^{n+1},$$

$$U_{j0}^{n+1}=g_1(r_j,t^{n+1}),\quad U_{jK}^{n+1}=g_2(r_j,t^{n+1}). \tag{15}$$

Here S_h denotes the discrete averaging operator

$$S_h(U_k^n) = \frac{2}{R^2} \sum_{j=0}^{J} \widetilde{r}_j U_{jk}^n h.$$

Note that equations (14) approximate the nonlocal flux conjugation conditions:

$$\sum_{j=0}^{J} \widetilde{r}_j \left(\frac{U_{0,K_1+1}^{n+1} - U_{0,K_1}^{n+1}}{H} - \frac{H}{2} \left(\frac{U_{0K_1}^{n+1} - U_{0K_1}^{n}}{\tau} - f_{0K_1}^{n+1} \right) \right) h \qquad (16)$$

$$= \sum_{j=0}^{J} \widetilde{r}_j \left(\frac{U_{0,K_1}^{n+1} - U_{j,K_1-1}^{n+1}}{H} + \frac{H}{2} \left(\frac{U_{0K_1}^{n+1} - U_{0K_1}^{n}}{\tau} - f_{0K_1}^{n+1} \right) \right) h.$$

For $U, V \in D_h$, such that $U_{j0} = U_{jK} = 0,\ V_{j0} = V_{jK} = 0,\ r_j \in \bar{\omega}_r$ the formulas

$$(U, V) = \sum_{j=0}^{J} \widetilde{r}_j \Big(\sum_{k=1}^{K_1-1} U_{jk} V_{jk} H + \sum_{k=K_2+1}^{K-1} U_{jk} V_{jk} H \Big) h + \frac{R^2}{2} \sum_{k=K_1}^{K_2} U_{0k} V_{0k} h,$$

$$\|U\| = (U, U)^{1/2}$$

define a scalar product and a norm in this vector space (in 2D discrete space !!!).

Let us define two operators for $U \in D_h$:

$$\mathcal{A}_1^h U = \begin{cases} A_1^h U_{\cdot,k}, & (r_j, z_k) \in \bar{\omega}_r \times (\omega_{z1} \cup \omega_{z3}), \\ 0, & z_k \in \bar{\omega}_{z2}, \end{cases}$$

$$\mathcal{A}_2^h U = \begin{cases} A_2^h U_{jk}, & (r_j, z_k) \in \bar{\omega}_r \times (\omega_{z1} \cup \omega_{z3}), \\ A_2^h U_{0k}, & z_k \in \omega_{z2}, \\ \frac{1}{H^2}\big(- S_h(U_{\cdot,K_1-1}) + 2U_{0K_1} - U_{0,K_1+1}\big), & k = K_1, \\ \frac{1}{H^2}\big(- S_h(U_{\cdot,K_2+1}) + 2U_{0K_2} - U_{0,K_2-1}\big), & k = K_2. \end{cases}$$

Then we can write the backward Euler scheme as

$$\frac{U^{n+1} - U^n}{\tau} + (\mathcal{A}_1^h + \mathcal{A}_2^h)U^{n+1} = F^{n+1}, \quad (r_j, z_k) \in \Omega_{h,RD}.$$

LEMMA

The discrete operators $\mathcal{A}_1^h$ and $\mathcal{A}_2^h$ are symmetric and non-negative and positive definite operators, respectively.

PROOF.

Applying the summation by part formula we get

$$(\mathcal{A}_1^h U, V) = \sum_{j=1}^{J} r_{j-\frac{1}{2}} \Big(\sum_{k=1}^{K_1-1} \partial_r U_{jk} \partial_r V_{jk} H + \sum_{k=K_2+1}^{K-1} \partial_r U_{jk} \partial_r V_{jk} H \Big) h$$
$$= (U, \mathcal{A}_1^h V).$$

Thus, $\mathcal{A}_1^h$ is symmetric and non-negative definite operator.

$$(\mathcal{A}_2^h U, V) = \sum_{j=0}^{J} \tilde{r}_j \Big(\sum_{k=1}^{K_1} \partial_z U_{jk}\, \partial_z V_{jk} H + \sum_{k=K_2+1}^{K} \partial_z U_{jk}\, \partial_z V_{jk} H \Big) h$$
$$+ \frac{R^2}{2} \sum_{k=K_1+1}^{K_2} \partial_z U_{0k}\, \partial_z V_{0k} H.$$

The nonlocal conjugation conditions (16) are used to derive these equalities.

ADI scheme

Then ADI scheme is written as

$$\frac{U^{n+\frac{1}{2}} - U^n}{\tau/2} + \mathcal{A}_1^h U^{n+\frac{1}{2}} + \mathcal{A}_2^h U^n = F^{n+1/2}, \ (r_j, z_k) \in \Omega_{h,RD}, \quad (17)$$

$$\frac{U^{n+1} - U^{n+\frac{1}{2}}}{\tau/2} + \mathcal{A}_1^h U^{n+\frac{1}{2}} + \mathcal{A}_2^h U^{n+1} = F^{n+1/2}. \quad (18)$$

Lemma

If U^n is the solution of ADI scheme (17)-(18)*, when $f^n \equiv 0$ and $g_1^n = g_2^n \equiv 0$, then the following stability estimate is valid*

$$\|(I + \frac{\tau}{2}\mathcal{A}_2^h)U^n\| \le \|(I + \frac{\tau}{2}\mathcal{A}_2^h)U^0\|. \quad (19)$$

Proof.

The ADI scheme (17)-(18) gives $U^{n+1} = RU^n$ with

$$R = \left(I + \frac{\tau}{2}\mathcal{A}_2^h\right)^{-1}\left(I - \frac{\tau}{2}\mathcal{A}_1^h\right)\left(I + \frac{\tau}{2}\mathcal{A}_1^h\right)^{-1}\left(I - \frac{\tau}{2}\mathcal{A}_2^h\right).$$

We rewrite this relations as

$$\left(I + \frac{\tau}{2}\mathcal{A}_2^h\right)U^{n+1} = \widetilde{R}\left(I + \frac{\tau}{2}\mathcal{A}_2^h\right)U^n.$$

By induction we prove that

$$\left(I + \frac{\tau}{2}\mathcal{A}_2^h\right)U^n = \widetilde{R}^n\left(I + \frac{\tau}{2}\mathcal{A}_2^h\right)U^0.$$

It follows from Lemma 2 that $\|\widetilde{R}\| \le 1$, thus

$$\|\widetilde{R}^n\| \le \|\widetilde{R}\|^n \le 1.$$

Due to nonlocal conjugation conditions the classical factorization algorithm should be modified in order to solve 1D subproblems (18).

LEMMA

The unique solution of the linear system of equations (18) *exists and it can be computed by using the efficient factorization algorithm.*

We get a linear system of two equations to find $U_{0K_1}^{n+1}$, $U_{0K_2}^{n+1}$:

$$\begin{cases} A_{11} U_{0K_1}^{n+1} + A_{12} U_{0K_2}^{n+1} = B_1 \\ A_{21} U_{0K_1}^{n+1} + A_{22} U_{0K_2}^{n+1} = B_2, \end{cases}$$

The accuracy of the reduced dimension model

We investigate the accuracy of the reduced dimension model (5)-(11). In the first example a domain with two nodes and one edge is used.
For the space discretization a uniform grid Ω_h with $J = 100$, $K = 1600$ is used, and integration in time is done with $\tau = 0.0005$.
Table 1 gives for a sequence of reduction parameter δ errors $e(\delta)$

$$e(\delta) = \max_{(r_j, z_k) \in \Omega_h h} \left| U_{jk}^N - U_{jk}^N(\delta) \right|$$

of the reduced dimension model

TABLE : Errors $e(\delta)$ of the discrete solution of the reduced dimension model (5)-(11) for a sequence of truncation parameter δ.

	$\delta = 0.05$	$\delta = 0.1$	$\delta = 0.15$	$\delta = 0.2$	$\delta = 0.25$
$e(\delta)$	0.2471	0.0377	0.0056	0.00083	0.00013

CPU time for computing the full model solution is 11.4 seconds, while for the reduced dimension model and $\delta = 0.25$ the time is reduced to 5.9 seconds, for $\delta = 0.1$ the CPU time is reduced to 2.5 seconds.

Example 2. In order to show the robustness of the proposed discrete scheme a domain with three full dimension nodes and two edges is considered. One additional node takes into account the influence of the source function.

For this test problem we get a linear system of four equations to find $U_{0K_1}^{n+1}$, $U_{0K_2}^{n+1}$, $U_{0K_3}^{n+1}$, $U_{0K_4}^{n+1}$. A matrix of this system is tridiagonal, thus the standard factorization algorithm is used to solve it.

TABLE : CPU time for computing the reduced dimension model (5)-(11) solution for a sequence of truncation parameter δ. The column δ^* gives CPU time for computing the full model solution.

	$\delta = \delta^*$	$\delta = 0.25$	$\delta = 0.2$	$\delta = 0.15$	$\delta = 0.1$
CPU time (δ)	24.9	17.0	14.6	12.2	9.8

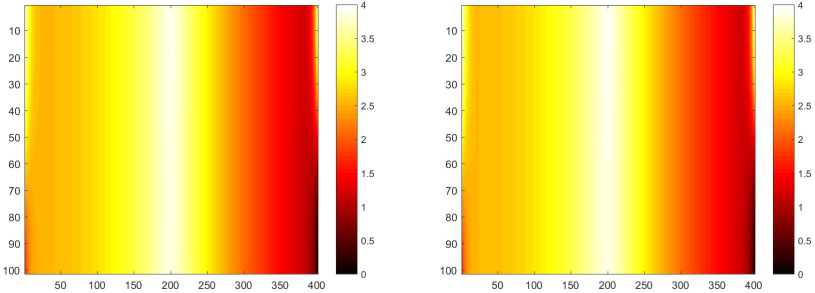

FIGURE : Full model and reduced dimension model with $\delta = 0.1$

Conclusions and Future Work

1.A finite volume method is used to approximate space differential operators with nonclassical conjugation conditions between the 3D and 2D parts.

2. The ADI scheme leads to non-iterative implementation algorithm and a set of one dimensional linear systems are solved by using the factorization algorithm. An efficient modification of the basic factorization algorithm is developed to resolve non-local conjugation conditions.

3. It is proved that the proposed discrete scheme is unconditionally stable.

4. **Future work**: more complicated geometry, parallel algorithms, Navier-Stokes model.