ADI SCHEME FOR PARTIALLY DIMENSION REDUCED HEAT CONDUCTION MODELS

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ADI scheme for dimension reduced model
Let us assume that initial and boundary conditions and all coefficients satisfy the radial symmetry condition, thus we get the following problem

\[
\frac{\partial u}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} + f(r, z, t), \quad (r, z, t) \in Q_T = \Omega \times (0, T],
\]

(1)

\[
u(r, 0, t) = g_1(r, t), \quad u(r, l, t) = g_2(r, t), \quad (r, t) \in (0, R] \times (0, T],
\]

(2)

\[
r \frac{\partial u}{\partial r} = 0, \quad 0 < z < l, \quad r = 0 \text{ and } r = R, \quad 0 < t \leq T,
\]

(3)

\[
u(r, z, 0) = u^0(r, z), \quad (r, z) \in \Omega.
\]

(4)
Let $S(u)$

$$S(u) = \frac{2}{R^2} \int_0^R ru(r, z, t) dr$$

denote the averaging operator.

We assume that the initial condition $u^0$ and source function $f$ satisfy the relations

$$u^0(r, z) = S(u^0), \quad f(r, z, t) = S(f), \quad (z, t) \in (0, l) \times (0, T].$$

It means that $u^0$ and $f$ do not depend on $r$ within the tube $T$

Denote a reduced tube $T_\delta = D \times (\delta, l - \delta)$ and $\Omega_\delta = \{(r, z) \in (0, R) \times (\delta, l - \delta)\}$. 
Function $U$ is called an approximate solution to problem (1) – (4) if it satisfies the following problem

\[
\frac{\partial U}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{\partial^2 U}{\partial z^2} + f(z, t), \quad (r, z, t) \in (\Omega \setminus \Omega_\delta) \times (0, T],
\]

\[
\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial z^2} + f(z, t), \quad (r, z, t) \in \Omega_\delta \times (0, T),
\]

\[
U(r, 0, t) = g_1(r, t), \quad U(r, l, t) = g_2(r, t), \quad (r, t) \in (0, R] \times (0, T],
\]

\[
r \frac{\partial U}{\partial r} = 0, \quad z \in (0, \delta) \cup (l - \delta, l), \quad r = 0, r = R, \quad 0 < t \leq T,
\]

\[
U(r, z, 0) = u^0(r, z), \quad (r, z) \in \Omega.
\]

In $\Omega_\delta \times (0, T]$ the solution $U$ do not depend on $r$, i.e.

\[
U(r, z, t) = S(U), \quad (r, z, t) \in \Omega_\delta \times (0, T].
\]
From the weak form of the heat equation, it follows that the conjugation conditions are valid at the truncations of the tube

\[ U\big|_{z=\delta-0} = U\big|_{z=\delta+0}, \quad U\big|_{z=l-\delta-0} = U\big|_{z=l-\delta+0}; \quad (10) \]

\[ \frac{\partial S(U)}{\partial z}\big|_{z=\delta-0} = \frac{\partial U}{\partial z}\big|_{z=\delta+0}, \quad \frac{\partial U}{\partial z}\big|_{z=l-\delta-0} = \frac{\partial S(U)}{\partial z}\big|_{z=l-\delta+0}. \quad (11) \]

The conditions (10) are classical and mean that \( U \) is continuous at the truncation points.

The remaining two conditions (11) are nonlocal and they define the conservation of full fluxes along the separation lines.
For functions defined on the grid $\Omega_h \times \omega_t$ we introduce the discrete operators with respect to $z$ and $r$:

$$
\partial_z U_{jk}^n := \frac{U_{jk}^n - U_{j,k-1}^n}{H}, \quad A_2^h U_{jk}^n := -\frac{1}{H} \left( \partial_z U_{j,k+1}^n - \partial_z U_{j,k}^n \right).
$$

$$
\partial_r U_{jk}^n := \frac{U_{jk}^n - U_{j-1,k}^n}{h}, \quad A_1^h U_{jk}^n := -\frac{1}{\tilde{r}_j h} \left( r_{j+\frac{1}{2}} \partial_r U_{j+1,k}^n - r_{j-\frac{1}{2}} \partial_r U_{j,k}^n \right),
$$

where

$$
\tilde{r}_0 = \frac{1}{8} h, \quad \tilde{r}_j = r_j, \quad 1 \leq j < J, \quad \tilde{r}_J = \frac{1}{2} \left( R - \frac{h}{4} \right), \quad r_{-\frac{1}{2}} = 0, \quad r_{J+\frac{1}{2}} = 0.
$$
Then the heat conduction problem (1)-(4) is approximated by the following Alternating Direction Implicit (ADI) scheme

\[
\frac{U_{jk}^{n+\frac{1}{2}} - U_{jk}^n}{\tau/2} + A_1^h U_{jk}^{n+\frac{1}{2}} + A_2^h U_{jk}^n = f_{jk}^{n+\frac{1}{2}}, \quad (r_j, z_k) \in \bar{\omega}_r \times \omega_z, \quad (12)
\]

\[
\frac{U_{jk}^{n+1} - U_{jk}^{n+\frac{1}{2}}}{\tau/2} + A_1^h U_{jk}^{n+\frac{1}{2}} + A_2^h U_{jk}^{n+1} = f_{jk}^{n+\frac{1}{2}}, \quad (r_j, z_k) \in \bar{\omega}_r \times \omega_z.
\]
**Lemma**

If a solution of the problem (1)-(4) is sufficiently smooth, then the approximation error of ADI scheme (12) is $O(\tau^2 + h^2 + H^2)$. 
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**Proof.**

The solution of ADI scheme (12) satisfies the scheme

$$
\frac{U_{jk}^{n+1} - U_{jk}^n}{\tau} + A_1^h \left( \frac{U_{jk}^{n+1} + U_{jk}^n}{2} \right) + A_2^h \left( \frac{U_{jk}^{n+1} + U_{jk}^n}{2} \right) \\
+ \frac{\tau^2}{4} A_1^h A_2^h \left( \frac{U_{jk}^{n+1} - U_{jk}^n}{\tau} \right) = f_{jk}^{n+\frac{1}{2}}.
$$

which is equivalent to the classical symmetrical finite difference scheme.
**Lemma**

The discrete operators $A^h_1$ and $A^h_2$ are symmetric and non-negative and positive definite operators, respectively.
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The discrete operators $A_1^h$ and $A_2^h$ are symmetric and non-negative and positive definite operators, respectively.

**Proof.**

Applying the summation by parts formula and taking into account the boundary conditions for vectors $u, v$ we get

$$
(A_2^h u, v) = \sum_{k=1}^{K-1} (A_2^h u)_k v_k H = (\partial_z u, \partial_z v).
$$

It follows that $A_2^h$ is symmetric. Since $A_2^h \phi_i = \lambda_i \phi_i$ has a complete set of eigenvectors $\phi_i$, $i = 1, \ldots, K - 1$, and all eigenvalues are positive $\lambda_i > 0$, $A_2^h$ is a positive-definite operator.
Proof.
Now we consider the operator $A_1^h$. Applying the summation by parts formula we get

$$[A_1^h u, v]_r = (\partial_r u, \partial_r v)_r.$$

It follows from the obtained equality, that $A_1^h$ is symmetric operator.
The eigenvalue problem $A_1^h \psi_l = \mu_l \psi_l$ has a complete set of eigenvectors $\psi_l, l = 0, \ldots, J$, one eigenvalue $\mu_0 = 0$ and the remaining eigenvalues are positive $\mu_l > 0$. 

☐
Lemma

ADI scheme (12) is unconditionally stable.
**Lemma**

**ADI scheme (12) is unconditionally stable.**

**Proof.**

The Fourier stability analysis is used. Let us consider the solution of ADI scheme (12) in the case when boundary conditions $g_j = 0$, $j = 1, 2$. Since operators $A^h_1$ and $A^h_2$ commute, the solution of (12) can be written as

$$
U_{jk}^{n+1} = \sum_{l=0}^{J} \sum_{r=0}^{K-1} c_{lr}^{n+1} \psi_l(r_j) \phi_r(z_k).
$$

Substituting this formula into equations (12) we obtain the stability equations for each mode

$$
c_{lr}^{n+1} = q_{lr} c_{lr}^n, \quad q_{lr} = \frac{(1 - 0.5\tau\lambda_r)(1 - 0.5\tau\mu_l)}{(1 + 0.5\tau\lambda_r)(1 + 0.5\tau\mu_l)}.
$$

Since eigenvalues $\lambda_r > 0$, $\mu_l \geq 0$, then the ADI scheme (12) is unconditionally stable in the $L_2$ norm.
The backward Euler scheme

The heat conduction problem (5)-(11) is approximated by the backward Euler scheme

\[
\frac{U^{n+1}_{jk} - U^n_{jk}}{\tau} + A_1^h U^{n+1}_{jk} + A_2^h U^{n+1}_{jk} = f^{n+1}_{jk}, \quad (r_j, z_k) \in \bar{\omega}_r \times (\omega_1 \cup \omega_3),
\]

(13)

\[
\frac{U^{n+1}_{0k} - U^n_{0k}}{\tau} + A_1^h U^{n+1}_{0k} = f^{n+1}_{0k}, \quad z_k \in \omega_{z2},
\]

\[
\frac{U^{n+1}_{0K_1} - U^n_{0K_1}}{\tau} + \frac{1}{H^2} \left( - S_h(U^{n+1}_{K_1-1}) + 2U^{n+1}_{0K_1} - U^{n+1}_{0,K_1+1} \right) = f^{n+1}_{0K_1},
\]

(14)

\[
\frac{U^{n+1}_{0K_2} - U^n_{0K_2}}{\tau} + \frac{1}{H^2} \left( - S_h(U^{n+1}_{K_2+1}) + 2U^{n+1}_{0K_2} - U^{n+1}_{0,K_2-1} \right) = f^{n+1}_{0K_2},
\]

\[
U^{n+1}_{j0} = g_1(r_j, t^{n+1}), \quad U^{n+1}_{jK} = g_2(r_j, t^{n+1}).
\]

(15)
Here $S_h$ denotes the discrete averaging operator

$$S_h(U^n_k) = \frac{2}{R^2} \sum_{j=0}^{J} \tilde{r}_j U^n_{jk} h.$$  

Note that equations (14) approximate the nonlocal flux conjugation conditions:

$$\sum_{j=0}^{J} \tilde{r}_j \left( \frac{U^{n+1}_{0,K_1+1} - U^{n+1}_{0,K_1}}{H} - \frac{H}{2} \left( \frac{U^{n+1}_{0K_1} - U^n_{0K_1}}{\tau} - f^{n+1}_{0K_1} \right) \right) h = \sum_{j=0}^{J} \tilde{r}_j \left( \frac{U^{n+1}_{0,K_1} - U^{n+1}_{j,K_1-1}}{H} + \frac{H}{2} \left( \frac{U^{n+1}_{0K_1} - U^n_{0K_1}}{\tau} - f^{n+1}_{0K_1} \right) \right) h.$$  

(16)
For $U, V \in D_h$, such that $U_{j0} = U_{jK} = 0$, $V_{j0} = V_{jK} = 0$, $r_j \in \bar{\omega}_r$ the formulas

\[
(U, V) = \sum_{j=0}^{J} \tilde{r}_j \left( \sum_{k=1}^{K_1-1} U_{jk} V_{jk} H + \sum_{k=K_2+1}^{K-1} U_{jk} V_{jk} H \right) h + \frac{R^2}{2} \sum_{k=K_1}^{K_2} U_{0k} V_{0k} h,
\]

\[
\|U\| = (U, U)^{1/2}
\]

define a scalar product and a norm in this vector space (in 2D discrete space !!!).
Let us define two operators for $U \in D_h$:

$$A_1^h U = \begin{cases} A_1^h U,.k, & (r_j, z_k) \in \bar{\omega}_r \times (\omega_{z1} \cup \omega_{z3}), \\ 0, & z_k \in \bar{\omega}_{z2}, \end{cases}$$

$$A_2^h U = \begin{cases} A_2^h U_{jk}, & (r_j, z_k) \in \bar{\omega}_r \times (\omega_{z1} \cup \omega_{z3}), \\ A_2^h U_{0k}, & z_k \in \omega_{z2}, \\ \frac{1}{H^2} \left( - S_h(U,.K_{1-1}) + 2U_0K_1 - U_{0,K_{1+1}} \right), & k = K_1, \\ \frac{1}{H^2} \left( - S_h(U,.K_{2+1}) + 2U_0K_2 - U_{0,K_{2-1}} \right), & k = K_2. \end{cases}$$
Then we can write the backward Euler scheme as

\[ \frac{U^{n+1} - U^n}{\tau} + (A_1^h + A_2^h)U^{n+1} = F^{n+1}, \quad (r_j, z_k) \in \Omega_{h, RD}. \]

**Lemma**

*The discrete operators $A_1^h$ and $A_2^h$ are symmetric and non-negative and positive definite operators, respectively.*
Proof.
Applying the summation by part formula we get

\[(A_1^h U, V) = \sum_{j=1}^{J} r_j \frac{1}{2} \left( \sum_{k=1}^{K_1-1} \partial_r U_{jk} \partial_r V_{jk} H + \sum_{k=K_2+1}^{K-1} \partial_r U_{jk} \partial_r V_{jk} H \right) h \]

\[= (U, A_1^h V). \]

Thus, \(A_1^h\) is symmetric and non-negative definite operator.

\[(A_2^h U, V) = \sum_{j=0}^{J} \tilde{r}_j \left( \sum_{k=1}^{K_1} \partial_z U_{jk} \partial_z V_{jk} H + \sum_{k=K_2+1}^{K} \partial_z U_{jk} \partial_z V_{jk} H \right) h \]

\[+ \frac{R^2}{2} \sum_{k=K_1+1}^{K_2} \partial_z U_{0k} \partial_z V_{0k} H.\]

The nonlocal conjugation conditions (16) are used to derive these equalities.
Then ADI scheme is written as

\[
\frac{U^{n+\frac{1}{2}} - U^n}{\tau/2} + A^h_1 U^{n+\frac{1}{2}} + A^h_2 U^n = F^{n+1/2}, \quad (r_j, z_k) \in \Omega_{h,RD},
\] (17)

\[
\frac{U^{n+1} - U^{n+\frac{1}{2}}}{\tau/2} + A^h_1 U^{n+\frac{1}{2}} + A^h_2 U^{n+1} = F^{n+1/2}.
\] (18)

**Lemma**

If \( U^n \) is the solution of ADI scheme (17)-(18), when \( f^n \equiv 0 \) and \( g^1_1 = g^2_2 \equiv 0 \), then the following stability estimate is valid

\[
\| (I + \frac{\tau}{2} A^h_2) U^n \| \leq \| (I + \frac{\tau}{2} A^h_2) U^0 \|.
\] (19)
**Proof.**

The ADI scheme (17)-(18) gives $U^{n+1} = RU^n$ with

$$ R = \left( I + \frac{\tau}{2} A_2^h \right)^{-1} \left( I - \frac{\tau}{2} A_1^h \right) \left( I + \frac{\tau}{2} A_1^h \right)^{-1} \left( I - \frac{\tau}{2} A_2^h \right). $$

We rewrite this relations as

$$ \left( I + \frac{\tau}{2} A_2^h \right) U^{n+1} = \tilde{R} \left( I + \frac{\tau}{2} A_2^h \right) U^n. $$

By induction we prove that

$$ \left( I + \frac{\tau}{2} A_2^h \right) U^n = \tilde{R}^n \left( I + \frac{\tau}{2} A_2^h \right) U^0. $$

It follows from Lemma 2 that $\|\tilde{R}\| \leq 1$, thus

$$ \|\tilde{R}^n\| \leq \|\tilde{R}\|^n \leq 1. $$
Due to nonlocal conjugation conditions the classical factorization algorithm should be modified in order to solve 1D subproblems (18).

**Lemma**

The unique solution of the linear system of equations (18) exists and it can be computed by using the efficient factorization algorithm.

We get a linear system of two equations to find \( U_{0K_1}^{n+1}, U_{0K_2}^{n+1} \):

\[
\begin{align*}
A_{11} U_{0K_1}^{n+1} + A_{12} U_{0K_2}^{n+1} &= B_1 \\
A_{21} U_{0K_1}^{n+1} + A_{22} U_{0K_2}^{n+1} &= B_2,
\end{align*}
\]
The accuracy of the reduced dimension model (5)-(11). In the first example a domain with two nodes and one edge is used.

For the space discretization a uniform grid \( \Omega_h \) with \( J = 100, K = 1600 \) is used, and integration in time is done with \( \tau = 0.0005 \). Table 1 gives for a sequence of reduction parameter \( \delta \) errors \( e(\delta) \)

\[
e(\delta) = \max_{(r_j,z_k) \in \Omega_h} \left| U^N_{jk} - U^N_{jk}(\delta) \right|
\]

of the reduced dimension model.
**Table:** Errors $e(\delta)$ of the discrete solution of the reduced dimension model (5)-(11) for a sequence of truncation parameter $\delta$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$e(\delta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.2471</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0377</td>
</tr>
<tr>
<td>0.15</td>
<td>0.0056</td>
</tr>
<tr>
<td>0.2</td>
<td>0.00083</td>
</tr>
<tr>
<td>0.25</td>
<td>0.00013</td>
</tr>
</tbody>
</table>

CPU time for computing the full model solution is 11.4 seconds, while for the reduced dimension model and $\delta = 0.25$ the time is reduced to 5.9 seconds, for $\delta = 0.1$ the CPU time is reduced to 2.5 seconds.
Example 2. In order to show the robustness of the proposed discrete scheme a domain with three full dimension nodes and two edges is considered. One additional node takes into account the influence of the source function.

For this test problem we get a linear system of four equations to find $U_{0K_1}^{n+1}, U_{0K_2}^{n+1}, U_{0K_3}^{n+1}, U_{0K_4}^{n+1}$. A matrix of this system is tridiagonal, thus the standard factorization algorithm is used to solve it.
**Table**: CPU time for computing the reduced dimension model (5)-(11) solution for a sequence of truncation parameter $\delta$. The column $\delta^*$ gives CPU time for computing the full model solution.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>CPU time ($\delta$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta^*$</td>
<td>24.9</td>
</tr>
<tr>
<td>$\delta = 0.25$</td>
<td>17.0</td>
</tr>
<tr>
<td>$\delta = 0.2$</td>
<td>14.6</td>
</tr>
<tr>
<td>$\delta = 0.15$</td>
<td>12.2</td>
</tr>
<tr>
<td>$\delta = 0.1$</td>
<td>9.8</td>
</tr>
</tbody>
</table>
**Figure**: Full model and reduced dimension model with $\delta = 0.1$
CONCLUSIONS AND FUTURE WORK

1. A finite volume method is used to approximate space differential operators with nonclassical conjugation conditions between the 3D and 2D parts.
2. The ADI scheme leads to non-iterative implementation algorithm and a set of one dimensional linear systems are solved by using the factorization algorithm. An efficient modification of the basic factorization algorithm is developed to resolve non-local conjugation conditions.
3. It is proved that the proposed discrete scheme is unconditionally stable.
4. **Future work**: more complicated geometry, parallel algorithms, Navier-Stokes model.