Stability analysis of finite difference schemes for parabolic type problems with nonlocal boundary conditions

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Outline

Problem Formulation

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Parabolic problem

Let us consider domain $D = (0, 1)$. We formulate an initial boundary value parabolic problem, when one boundary condition is nonlocal:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in D, \ t > 0, \tag{1}$$

$$-\frac{\partial u}{\partial x}\Big|_{x=0} = \mu_0(t), \quad au(1,t) + \int_0^1 u(x,t)\,dx = \mu_1(t), \quad t > 0,$$

$$u(x,0) = \nu(x), \quad x \in \bar{D} := [0,1],$$

where f, φ, μ_0, μ_1 are known functions, and a is a given constant.

Pseudoparabolic Problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial t \partial x}\left(g(x)\frac{\partial u}{\partial x}\right) - q(x,t)u + f(x,t),$$

$$u(0,t) = \mu_0(t), \qquad (2)$$

$$k(x)\frac{\partial u}{\partial x}\Big|_{x=1} + \frac{\partial}{\partial t}\left(g(x)\frac{\partial u}{\partial x}\right)\Big|_{x=1} = \int_0^1 r(x)u(x,t)\,dx + \mu_1(t),$$

$$u(x,0) = \nu(x), \quad x \in \bar{D},$$

where k, g, r, μ_0, μ_1 are known functions, such that

$$0 < k_m \le k(x) \le k_M, \;\; 0 < g_m \le g(x) \le g_M, \;\; 0 \le r(x) \le r_M, \;\; q(x) \ge 0.$$

General Remarks on Stability Analysis

- Superposition of solutions of classical BVP (for stationary problems). The Green function method for parabolic problems?

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- Superposition of solutions of classical BVP (for stationary problems). The Green function method for parabolic problems?
- The maximum principle for parabolic problems. But what about pseudoparabolic problems?
- The energy estimates in different norms. They can be applied for general equations with non-constant coefficients, but this method gives only sufficient stability conditions. How to find the right norm or functional?

The Eigenvalue Criterion for Non-Normal Matrices

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Let assume that $\max_j \text{Re}\,\lambda_j \leq \omega$, then

$$\|w(t)\| \leq \|U\|\,\|U^{-1}\|\,e^{t\omega}\|w(0)\|.$$

The Finite Difference Scheme

The considered domain $\bar{D}$ is covered by the uniform mesh

$$\bar{D}_h = \{x_j : \ \ x_j = jh, \ \ j = 0, \dots, J\},$$

$x_J = b_x$, $\bar{D}_h = D_h \cup \partial D_h$.

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Let ω_τ be a uniform time mesh

$$\omega_\tau = \{t^n : \ \ t^n = n\tau, \quad n = 0, \dots, N, \ N\tau = T\},$$

where τ is the time step.

The following notations for difference operators are used

$$\partial_x U_j^n = (U_{j+1}^n - U_j^n)/h, \quad \partial_{\bar{x}} U_j^n = (U_j^n - U_{j-1}^n)/h,$$

$$\partial_x^2 U_j^n = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}, \quad \partial_{\bar{t}} U_j^n = \frac{U_j^n - U_j^{n-1}}{\tau},$$

$$S_h U := \frac{h}{2}(U_0 + U_J) + \sum_{j=1}^{J-1} U_j h.$$

The backward Euler finite difference scheme is defined as

$$\partial_{\bar t} U_j^n = \partial_x^2 U_j^n + f_j^n, \quad x_j \in D_h, \ \ n > 0, \tag{3}$$

$$- \partial_x U_0^n + \frac{h}{2} (\partial_{\bar t} U_0^n - f_0^n) = \mu_0(t^n),$$

$$a U_J^n + S_h U^n = \mu_1(t^n), \quad n > 0,$$

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&U_j^0 = \nu(x_j), \quad x_j \in \bar{D}_h.
\end{aligned} \tag{3}$$

We restrict to the analysis of the case $a \geq 0$.

At each time moment t^n a solution of problem (3) can be computed by the modified factorization algorithm.

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Since $a \geq 0$, then the discrete problem has a unique solution for $t^n \in \omega_\tau$.

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Let us denote the error function $Z_j^n = U_j^n - u(x_j, t^n)$.

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It satisfies the discrete problem

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\begin{aligned}
& \partial_{\bar{t}} Z_j^n = \partial_x^2 Z_j^n + \psi_j^n, \quad x_j \in D_h, \quad n > 0, \\
& -\partial_x Z_0^n + \frac{h}{2}\partial_{\bar{t}} Z_0^n = \psi_0^n, \quad a Z_J^n + S_h Z^n = \psi_J^n, \quad n > 0, \\
& Z_j^0 = 0, \quad x_j \in \bar{D}_h,
\end{aligned}
\tag{4}
$$

where ψ_j^n is the approximation error of the finite difference scheme (3).

For sufficiently smooth solutions of the differential problem (1) the approximation errors can be estimated as

$$|\psi_j^n| \leq C(\tau + h^2), \quad j = 0, \ldots, J.$$

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Let us assume that additional compatibility conditions are satisfied for the solution of problem (1) and the approximation errors of the boundary condition satisfy the estimate

$$|\partial_{\bar{t}} \psi_j^n| \leq C(\tau + h^2), \quad j = 0, J.$$

In order to get homogeneous boundary conditions, we construct the auxiliary discrete function

$$V_j^n = Ax_j^2 + Bx_j, \quad x_j \in \bar{D}_h,$$
$$A = \frac{\psi_J^n + (a+1/2)\psi_0^n}{a(1-h) + (2+h^2)/6 - h/2}, \quad B = -Ah - \psi_0^n,$$

and introduce a new error function $W_j^n = Z_j^n - V_j^n$.

W_j^n satisfies the discrete problem with homogeneous boundary conditions

$$\partial_{\bar t} W_j^n = \partial_x^2 W_j^n + \varphi_j^n, \quad x_j \in D_h, \quad n > 0, \tag{5}$$

$$-\partial_x W_0^n + \frac{h}{2}\partial_{\bar t} W_0^n = 0, \quad aW_J^n + S_h W^n = 0, \quad n > 0,$$

$$W_j^0 = -V_j^0, \quad x_j \in \bar D_h,$$

where $\varphi_j^n = \psi_j^n - \partial_{\bar t} V_j^n + \partial_x^2 V_j^n$.

We introduce a summation operator of a mesh function U:

$$S_{j+\frac{1}{2}}U := \sum_{\ell=1}^{j} U_\ell h + \frac{h}{2}U_0, \quad j = 0, \dots, J-1.$$

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Some mesh counterparts of the inner products and norms:

$$(U, V) = \frac{h}{2}U_0V_0 + \sum_{j=1}^{J-1} U_jV_j\, h + \frac{h}{2}U_JV_J, \quad \|U\| = \sqrt{(U, U)},$$

$$(SU, SV)_2 = \sum_{j=0}^{J-1} S_{j+\frac{1}{2}}U\, S_{j+\frac{1}{2}}V\, h, \quad \|SU\|_2 = \sqrt{(SU, SU)_2}.$$

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The following imbedding inequality holds

$$\|SU\|_2 \leq \frac{1}{\sqrt{2}}\,\|U\|.$$

Multiplying FDS equations by h, summing up the obtained equations for $j = 1, \ldots, \ell$ and taking into account the first boundary condition, we get the mass balance equation

$$\partial_{\bar{t}} S_{\ell+\frac{1}{2}} W^n = \partial_x W_\ell^n + S_{\ell+\frac{1}{2}} \varphi^n. \tag{6}$$

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Taking the inner product of equation (6) and function SW^n, we get

$$(\partial_{\bar{t}} SW^n, SW^n)_2 = (\partial_x W^n, SW^n)_2 + (S\varphi^n, SW^n)_2. \tag{7}$$

Applying a formula of summation by parts we obtain

$$(\partial_x W^n, SW^n)_2 = \sum_{j=0}^{J-1} (W_{j+1}^n - W_j^n) S_{j+\frac{1}{2}} W^n$$

$$= -\sum_{j=1}^{J-1} W_j^n \frac{S_{j+\frac{1}{2}} W^n - S_{j-\frac{1}{2}} W^n}{h} h + W_J^n S_{J-\frac{1}{2}} W^n - W_0^n S_{\frac{1}{2}} W^n.$$

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Since $S_{\frac{1}{2}} W^n = \frac{h}{2} W_0^n$ and $S_{J-\frac{1}{2}} W^n = S_h W^n - \frac{h}{2} W_J^n$, due to the nonlocal boundary condition, the following equality is valid

$$(\partial_x W^n, SW^n)_2 = -\|W^n\|^2 - a\,(W_J^n)^2.$$

The first term can be rewritten as

$$(\partial_{\bar{t}} SW^n, SW^n)_2 = \frac{1}{2}\partial_{\bar{t}}\|SW^n\|_2^2 + \frac{\tau}{2}\|\partial_{\bar{t}} SW^n\|_2^2.$$

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The last term of (7) can be estimated as

$$(S\varphi^n, SW^n)_2 \leq \|S\varphi^n\|_2 \, \|SW^n\|_2 \leq \|S\varphi^n\|_2^2 + \frac{1}{4}\,\|SW^n\|_2^2.$$

We get the following stability estimate

$$\partial_{\bar{t}} \|SW^n\|_2^2 + \tau \|\partial_{\bar{t}} SW^n\|_2^2 + \frac{7}{2}\|SW^n\|_2^2 + 2a\,(W_J^n)^2 \le 2\|S\varphi^n\|_2^2.$$

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The method of bounding functions implies that

$$\|SW^n\|_2^2 \le \|SV^0\|_2^2 + \frac{4}{7}\max_{1\le k\le n} \|S\varphi^k\|_2^2.$$

FDS for pseudo-parabolic problem

We approximate problem (2) by the backward Euler finite difference scheme

$$\partial_{\bar t} U_j^n = \partial_x F_{j-\frac{1}{2}} U^n - q_j^n U_j^n + f_j^n, \quad x_j \in D_h, \ \ n > 0, \qquad (8)$$

$$F_{j-\frac{1}{2}} U^n := k_{j-\frac{1}{2}} \partial_{\bar x} U_j^n + \partial_{\bar t} \left(g_{j-\frac{1}{2}} \partial_{\bar x} U_j^n \right),$$

$$U_0^n = \mu_0^n, \quad F_{J-\frac{1}{2}} U^n + \frac{h}{2} (\partial_{\bar t} U_J^n + q_J^n U_J^n - f_J^n) = S_h U^n + \mu_1^n,$$

$$U_j^0 = \nu(x_j), \ x_j \in \bar D_h, \quad S_h U := \frac{h}{2}(r_0 U_0 + r_J U_J) + \sum_{j=1}^{J-1} r_j U_j h.$$

Results of General Stability Theory

Let us consider a template of 2 stage FDS:

$$(I + \tau R)\,\partial_{\bar{t}} U^n + AU^{n-1} = 0, \quad U^0 = \phi,$$

where $I + \tau R > 0$. Operator R is called a regularizator.
A sufficient stability condition is given by

$$R \geq \sigma_0 A, \quad \sigma_0 = \frac{1}{2} - \frac{1}{\tau \|A\|}.$$

Let us consider the heat conduction equation with

$$AU = -\partial_x \partial_{\bar{x}} U, \quad R = A.$$

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2. Symmetrical Euler scheme can be written as

$$\left(I + \tau \frac{1}{2} A\right) \partial_{\bar{t}} U^n + AU^{n-1} = 0.$$

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2. Symmetrical Euler scheme can be written as

$$\left(I + \tau \frac{1}{2} A\right) \partial_{\bar{t}} U^n + AU^{n-1} = 0.$$

3. Backward Euler scheme can be written as

$$(I + \tau A)\, \partial_{\bar{t}} U^n + AU^{n-1} = 0.$$

For the pseudoparabolic problem

$$\partial_{\bar{t}} U^n + \mu \partial_{\bar{t}} A U^n + A U^n = 0$$

the regularizator is equal to

$$R = \left(\frac{\mu}{\tau} + 1 \right) A.$$

For the pseudoparabolic problem

$$\partial_{\bar{t}} U^n + \mu \partial_{\bar{t}} A U^n + A U^n = 0$$

the regularizator is equal to

$$R = \Big(\frac{\mu}{\tau} + 1\Big) A.$$

The Fourier coefficients are defined as

$$c_j^n = \frac{1 + \mu\lambda_j}{1 + (\mu + \tau)\lambda_j}\, c_j^{n-1}.$$

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The Fourier coefficients are defined as

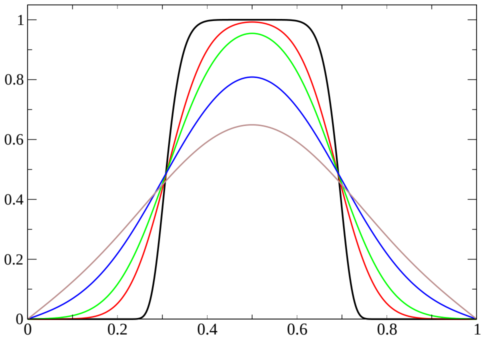
$$c_j^n = \frac{1 + \mu \lambda_j}{1 + (\mu + \tau) \lambda_j} \, c_j^{n-1}.$$

What about the case $\lambda_j < 0$?

Parabolic problem

$$\partial_{\bar{t}} U^n + AU^n = 0, \quad \lambda_j > 0,$$

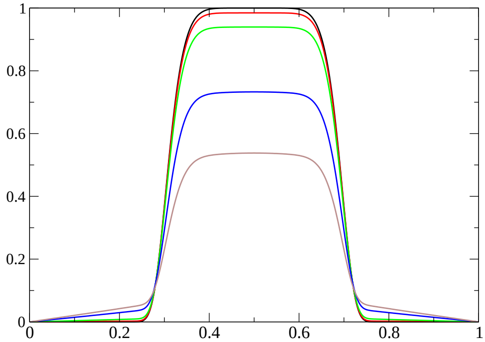
results for time moments $t = 0,\ 0.002,\ 0.004,\ 0.01,\ 0.02$.



Pseudoparabolic problem

$$\partial_{\bar{t}} U^n + 0.5 \partial_{\bar{t}} A U^n + A U^n = 0, \quad \lambda_j > 0,$$

results for time moments $t = 0,\ 0.005,\ 0.02,\ 0.1,\ 0.2.$



The error function $Z_j^n = U_j^n - u(x_j, t^n)$ satisfies the following discrete problem

$$\partial_{\bar{t}} Z_j^n = \partial_x F_{j-\frac{1}{2}} Z^n - q_j^n Z_j^n + \psi_j^n, \quad x_j \in D_h, \ \ n > 0, \tag{9}$$

$$Z_0^n = 0, \quad F_{J-\frac{1}{2}} Z^n + \frac{h}{2}\left(\partial_{\bar{t}} Z_J^n + q_J^n Z_J^n\right) = S_h Z^n + \psi_J^n, \quad n > 0,$$

$$Z_j^0 = 0, \quad x_j \in \bar{D}_h.$$

We introduce some mesh counterparts of the inner products and norms:

$$(p\partial_{\bar{x}} U, \partial_{\bar{x}} V] = \sum_{j=1}^{J} p_{j-\frac{1}{2}} \partial_{\bar{x}} U_j \, \partial_{\bar{x}} V_j \, h, \quad \| U \|_\infty = \max_{0 \le j \le J} |U|.$$

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If $U_0 = 0$ and function $p(x) \geq p_m > 0$, then the following imbedding inequalities hold

$$\|U\|_\infty^2 \leq C_1(p)\,(p\partial_{\bar{x}}U, \partial_{\bar{x}}U], \quad C_1(p) = \sum_{j=1}^{J} \frac{h}{p_{j-\frac{1}{2}}}, \tag{10}$$

$$\|U\|^2 \leq C_2(p)\,(p\partial_{\bar{x}}U, \partial_{\bar{x}}U], \; C_2(p) = \sum_{j=1}^{J-1} h \sum_{l=1}^{j} \frac{h}{p_{l-\frac{1}{2}}} + \frac{h}{2}\sum_{l=1}^{J} \frac{h}{p_{l-\frac{1}{2}}},$$

$$\|U\| \leq \|U\|_\infty.$$

Taking the inner product of equation (9) and function Z^n, we get

$$\sum_{j=1}^{J-1} Z_j^n \partial_{\bar{t}} Z_j^n h = \sum_{j=1}^{J-1} Z_j^n \left(\partial_x F_{j-\frac{1}{2}} Z^n - q_j Z_j^n + \psi_j^n \right) h. \qquad (11)$$

Applying a formula of summation by parts and using boundary conditions we obtain

$$\sum_{j=1}^{J-1} Z_j^n \partial_x F_{j-\frac{1}{2}} Z^n h = -(k\partial_{\bar{x}} Z^n, \partial_{\bar{x}} Z^n] - (g\partial_{\bar{t}}\partial_{\bar{x}} Z^n, \partial_{\bar{x}} Z^n]$$

$$+ Z_J^n F_{J-\frac{1}{2}} Z^n = -(k\partial_{\bar{x}} Z^n, \partial_{\bar{x}} Z^n] - (g\partial_{\bar{t}}\partial_{\bar{x}} Z^n, \partial_{\bar{x}} Z^n]$$

$$- \frac{h}{2}\left(Z_J^n \partial_{\bar{t}} Z_J^n + q_J^n (Z_J^n)^2\right) + Z_J^n S_h Z_J^n + Z_J^n \psi_J^n. \qquad (12)$$

Putting (12) into (11) after simple computations we get the estimate

$$\frac{1}{2}\partial_{\bar t}\left(\|Z^n\|^2+(g\partial_{\bar x}Z^n,\partial_{\bar x}Z^n]\right)+\frac{\tau}{2}\left(\|\partial_{\bar t}Z^n\|^2+(g\partial_{\bar t}\partial_{\bar x}Z^n,\partial_{\bar t}\partial_{\bar x}Z^n]\right)$$
$$+\,(k\partial_{\bar x}Z^n,\partial_{\bar x}Z^n]+\|\sqrt{q^n}Z^n\|^2\le\varepsilon_1\|Z^n\|^2+\frac{1}{4\varepsilon_1}\|\widetilde{\psi}^n\|^2$$
$$+\,\|Z^n\|_\infty^2\left(S_h r+\varepsilon_2\right)+\frac{|\psi_J^n|^2}{4\varepsilon_2}, \tag{13}$$

where $\widetilde{\psi}_j^n=\psi_j^n$, for $j=1,\dots,J-1$, and $\widetilde{\psi}_j^n=0$, for $j=0,J$.

First, let us assume that

$$S_h r + \varepsilon_1 + \varepsilon_2 \leq \frac{\nu}{C_1(k)}, \quad \nu < 1.$$

Then we have the estimate

$$\partial_{\bar{t}} \left(\|Z^n\|^2 + (g\partial_{\bar{x}} Z^n, \partial_{\bar{x}} Z^n] \right) + 2(1-\nu)(k\partial_{\bar{x}} Z^n, \partial_{\bar{x}} Z^n]$$

$$\leq \frac{1}{2\varepsilon_1} \|\widetilde{\psi}^n\|^2 + \frac{|\psi_J^n|^2}{2\varepsilon_2}.$$

Using the imbedding inequalities (10) the following stability estimate is proved

$$\partial_{\bar{t}} \left(\|Z^n\|^2 + (g \partial_{\bar{x}} Z^n, \partial_{\bar{x}} Z^n] \right) + \frac{2(1-\nu) C_3}{1 + C_2 C_3}$$
$$\times \left(\|Z^n\|^2 + (g \partial_{\bar{x}} Z^n, \partial_{\bar{x}} Z^n] \right) \leq \frac{1}{2\varepsilon_1} \|\widetilde{\psi}^n\|^2 + \frac{|\psi_J^n|^2}{2\varepsilon_2},$$

where $C_3(k,g) = \min_{0 \leq x \leq 1} \frac{k(x)}{g(x)}$.

The method of bounding functions implies that

$$\|Z^n\|^2 + (g\partial_{\bar{x}} Z^n, \partial_{\bar{x}} Z^n] \le \frac{1 + C_2 C_3}{2(1-\nu)C_3} \max_{1 \le l \le n} \left(\frac{1}{2\varepsilon_1} \|\widetilde{\psi}^l\|^2 + \frac{|\psi_J^l|^2}{2\varepsilon_2} \right).$$

Let us consider the second case, when

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$$\begin{aligned}
&\partial_{\bar t}\left(\|Z^n\|^2 + (g\partial_{\bar x} Z^n, \partial_{\bar x} Z^n]\right) \\
&\quad \leq 2C_4(k\partial_{\bar x} Z^n, \partial_{\bar x} Z^n] + \frac{1}{2\varepsilon_1}\|\widetilde{\psi}^n\|^2 + \frac{|\psi_J^n|^2}{2\varepsilon_2} \\
&\quad \leq C_5\left(\|Z^n\|^2 + (g\partial_{\bar x} Z^n, \partial_{\bar x} Z^n]\right) + \frac{1}{2\varepsilon_1}\|\widetilde{\psi}^n\|^2 + \frac{|\psi_J^n|^2}{2\varepsilon_2}. \qquad (14)
\end{aligned}$$

Let us take a sufficiently small time step τ, such that $\tau C_5 < 1$. We denote by

$$q := \frac{1}{1 - \tau C_5} = e^{\tau C_6}.$$

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From the discrete analog of the Gronwall Lemma we deduce that

$$\|Z^n\|^2 + (g\partial_{\bar{x}} Z^n, \partial_{\bar{x}} Z^n]) \le \frac{1}{C_5}\left(e^{C_6 t^n} - 1\right)$$

$$\times \max_{1 \le \ell \le n}\left(\frac{1}{2\varepsilon_1}\|\widetilde{\psi}^\ell\|^2 + \frac{|\psi_J^\ell|^2}{2\varepsilon_2}.\right)$$

Thank You for Your attention!