Lecture 9

Popular tests of convergence of series.

1. D'Alembert test of convergence

(ratio test)

We restrict to the series with positive elements

$$\sum_{n=1}^{\infty} a_n, \qquad a_n > 0. \qquad (1)$$

(a generalization can be done for analysis of absolute convergence of general series with $a_n \geq 0$).

The criterium:

1. If
$$\boxed{\frac{a_{n+1}}{a_n} < 1, \quad \forall n > n_0 \ldots}$$
then the series (1) converges

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1.$$

A more general condition
$$\boxed{\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1.}$$

2. If $\forall n > N$

$$\frac{a_{n+1}}{a_n} > 1 \qquad L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$$

then the series diverges.

A more general condition

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$$

then the series (1) diverges

None conclusion can be done of (the test is inconclusive)

$$\boxed{L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1}$$

<u>Examples</u> $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{n}{n+1} = \underline{1} .$$

This series diverges

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1$$

This series converges!

Examples

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

Applying the ratio test we compute the limit

$$L = \lim_{n\to\infty} \left(\frac{n+1}{e^{n+1}} \cdot \frac{e^n}{n}\right) = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{1}{e} < 1$$

The series converges

$$\sum_{n=1}^{\infty} \frac{e^n}{n}$$

$$\lim_{n\to\infty} \frac{e^{n+1}}{e^n} \cdot \left(\frac{n}{n+1}\right) = e > 1$$

Thus the series diverges.

The Cauchy root test

$\sum_{n=1}^{\infty} a_n$ a given series

If $\forall n \geq n_0 \quad \sqrt[n]{a_n} < 1$,

then the series converges.

$$C = \lim_{n\to\infty} \sqrt[n]{a_n} \;; \quad C < 1.$$

If $\forall n \geq n_0$:

$$\sqrt[n]{a_n} > 1 \qquad \left(C = \lim_{n\to\infty} \sqrt[n]{a_n} > 1 \right)$$

then the series diverges.

If $C = 1$ then the test is inconclusive.

Try to apply the Cauchy test for $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ series.

Example.

$$\sum_{n=1}^{\infty} \frac{2^n}{n^6}$$

$$C = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^6}} = \lim_{n\to\infty} \frac{2}{(n^{1/n})^6} = 2$$

$C = 2 > 1$, the series diverges.

<u>Example</u>

$$\sum_{n=0}^{\infty} \frac{1}{2^{\lfloor n/2 \rfloor}} = 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \dots$$

$$a_{2n} = \frac{1}{2^n}$$

$$\lim_{n\to\infty} \sqrt[2n]{a_{2n}} = \frac{1}{\sqrt{2}} < 1$$

The series converges

Proof of the convergence of a series $\sum_{n=1}^{\infty} a_n$ if $C < 1$.

The proof is based on the comparison test

If $\forall n \geq n_0$ we have

$$\sqrt[n]{a_n} \leq C < 1, \quad \text{then}$$

$$a_n \leq C^n < 1 \ (\text{since } C < 1)$$

Since the geometric series $\sum_{n=n_0}^{\infty} C^n$ converges, then by the comparison test $\sum_n a_n$ ~~is~~ also converges.