Introduction: examples of algorithms and basics of complexity analysis

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1. Textbook (see references)

R. Čiegis. Duomenų struktūros, algoritmai ir jų analizė. Vilnius, "Technika", 2007.

2. Textbook (with pdf, see references)

T. Cormen, C. Leiserson, R. Rivest, C. Stein. Introduction to algorithms. The MIT Press, 2009.

3. Textbook (with pdf, see references)

T. Cormen. Algorithms unlocked. The MIT Press, 2013.

3. Video materials:

https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-006-introduction-to-algorithms-fall-2011/lecture-videos/

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It is easy to find many interesting examples around us.

1. You want to prepare a special pizza for a lunch.

We take a book of culinary recipes, it describes all steps, that should be done and in a specific sequence. So we have an algorithm for cooking a tasty pizza.

2. You need to pass the exam on "Theory of Algorithms". .

The corresponding algorithm can be used:

a) perform all laboratory work tasks,

b) solve the homework tasks,

c) pass the intermediate exam,

d) get positive evaluation of the session exam.

Everything looks simple and clear, just execute this algorithm in given order.

In this course, you will systematize the knowledge you already have on various algorithms and will learn many new useful things:

a) how to create a new algorithm for the selected problem;

b) how to effectively implement algorithms (even a very good algorithm can be underestimated if not implemented properly);

c) you will become familiar with algorithms for solving important applied problems including search of information, data sorting, transport logistics and complicated design tasks.

Where are algorithms used?

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We mention only few domains:

big data and analysis,
virtual reality,
artificial inteligence (AI),
artificial neural networks, machine learning,
internet of things,
digital audio, visual and video information: storage and transmission,
medicine,
robotics,
cryptocurrencies,
computer games ...

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A more precise definition is obtained when the computing device is presented as the famous Turing machine. The latter theoretical machine was created, when Turing provided a constructive definition of the algorithm.

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How long will it take for a malicious program to "hack" your bank account password?

Is information about cryptocurrencies securely protected in the blockchain?

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- The size of the matrix A with m rows and n columns, is equal to mn,
- Let us take a graph $G = (V, E)$, for which the number of vertices V is n, and the size of the edge set E is m, then a total number of data is equal to $(m + n)$.

Although the number of data characterizes the size of the task, it does not yet describe the complexity of the algorithm completely. Let's examine two important operations: addition of two matrices $A + B$ and multiplication of two matrices AB.

Let us assume that we have square matrices of size $n \times n$.

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Let us consider the matrix addition $A + B$

$$C = A + B, \quad C = (c_{ij}), \;\; 1 \leqslant i, j \leqslant n\,,$$
$$c_{ij} = a_{ij} + b_{ij}\,.$$

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This number of operations is of the same order as the number of initial data, since the number of coefficients for both matrices is $2n^2$.

Next let us consider the multiplication of two matrices

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad 1 \leqslant i, j \leqslant n \,.$$

In total we compute n^3 multiplication and $n^2(n-1)$ addition operations, or $(2n^3 - n^2)$ arithmetical operations.

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The complexity of both operations is estimated by using the same metric – the number of arithmetical operations.

However, not all algorithms are suitable for this measure.
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The complexity of any algorithm is equal to the number of basic operations of this algorithm.

Then the size of the problem is equal to the complexity of the known best algorithm used to solve it.

Let us recall some simple mathematical results, which greatly facilitate the analysis of algorithms.

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Very often the same algorithm is used to solve a given problem with various initial data.

For example, a company re-sorts information (data records) about their customers every day. As we will see data retrieval is in particular efficient when the data is already sorted.

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Let D_n be a set of all cases of initial data.
There are n elements in each collection d_m:

$$D_n = \{d_m : \ d_m = (a_1, a_2, \ldots, a_n), \quad a_j \in A, \ \ j = 1, 2, \ldots, n\}.$$

Denote by $T(d_m)$ a complexity of the algorithm for a specific collection of data d_m.

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Denote by $T(d_m)$ a complexity of the algorithm for a specific collection of data d_m.

The best case complexity is given by

$$T_G(n) = \min_{d_m \in D_n} T(d_m) \, .$$

2. The worst case complexity

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The complexity of the worst case is defined as

$$T_B(n) = \max_{d_m \in D_n} T(d_m) \, .$$

3. The average complexity

The given algorithm may be inefficient for some types for data, but often such sets are very few. Therefore, the most important information for most users is to know how long it takes on average to wait for the result. Such information allows a rational usage of human and hardware resources.

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Examples of such estimates will be given for most algorithms presented in our lectures.

Basic methods for development of algorithms

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The good news is that only a few basic methods exist for development of efficient algorithms. Thus we always have a short list of starting points.

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But You will see that parctically all algorithms in Top 10 list are developed by using these methods.

I hope, that this note is already a strongly motivating argument.

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We should answer the following two questions:

1. How to divide the task into a finite number of smaller tasks?
2. How to reduce the number of smaller tasks, since the direct analysis of all smaller tasks may fail even with the fastest supercomputers.

Testing of all cases

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John and Peter are good friends and they met today. OK, the last their meeting was quite a long time ago...

During conversation Peter explained that this day is very special for him, because all three his children – Inga, Julia and Justin – celebrate their birthdays. Peter suggested to John, who is a professional mathematician, to guess the age of each child.

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New Peter's remark was short – the eyes of the eldest child are blue. Upon learning this, John immediately told the age of each child!

From the first condition we learn, that the product of the ages of three children equals 36. It is easy to verify that there are only eight different options when this condition is met. They are given in the table.

TABLE: Eight options, when the product equals 36

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
Inga	1	1	1	1	1	2	2	3
Julia	1	2	3	4	6	2	3	3
Justin	36	18	12	9	6	9	6	4

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We compute the sums of ages for each option:

$$
\begin{aligned}
1+1+36 &= 38, & 1+2+18 &= 21, \\
1+3+12 &= 16, & 1+4+\ \ 9 &= 14, \\
1+6+\ \ 6 &= 13, & 2+2+\ \ 9 &= 13, \\
2+3+\ \ 6 &= 11, & 3+3+\ \ 4 &= 10.
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Now it is clear that this sum of years is equal to 13, because in all other cases John would already know the age of each child.

Two options remain $(1, 6, 6)$ and $(2, 2, 9)$.

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Since Justin is the eldest child only in the second case (that his eyes are blue, of course, does not matter), we conclude that Peter is raising twins Inga and Julia, who have turned two years old, and nine-year-old son Justin.

Security of information, passwords

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The secrecy and security of communications is currently a very important task and its importance will only increase in the future. Unauthorized reading of information and hybrid attacks on data centers grows also fastly.

Security of modern public key cryptographic algorithms is based on mathematical results for some very classical number theory problems. The state of the art in this field still is such, that the solution of these problems is possible only after a complete re-selection of all cases, and the total number of different options is so high that it is not possible to "crack" the password while the information is still relevant.

The Recursion Method

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The recurrence definition can be given as:

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It follows from this definition that

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \, .$$

Fibonacci numbers. Fibonacci numbers are a sequence of numbers where every number is the sum of the preceding two numbers.
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Task 1. Compute f_4 value.

Task 2. This task is more complicated: compute f_{30} and f_{31} values. What conclusion You can make?

We will show how recursion helps to solve logical games and puzzles.

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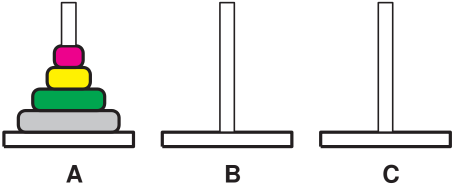
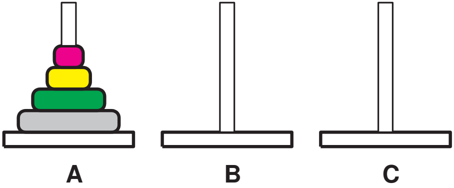
The Towers of Hanoi is a mathematical game or puzzle consisting of three rods and a number of disks of various diameters, which can slide onto any rod.
The puzzle begins with the disks stacked on one rod in order of decreasing size, the smallest at the top, thus approximating a conical shape. The objective of the puzzle is to move the entire stack to the last rod, obeying the following rules:

1. Only one disk may be moved at a time.

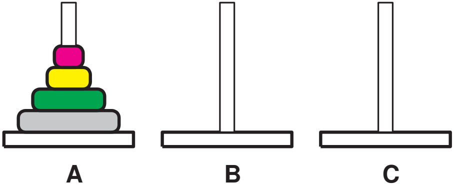
2. Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack or on an empty rod.

3. No disk may be placed on top of a disk that is smaller than it.


A
B
C



We can solve this problem by applying the following strategy:



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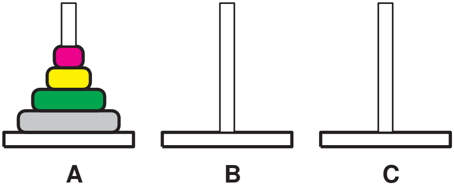
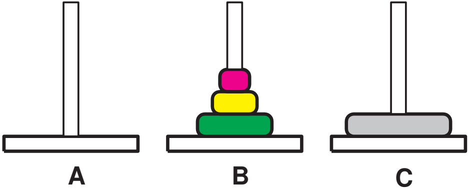
First, we move the $(n-1)$ top discs of A on the B rod, and we use the empty rod C as an auxiliary one.



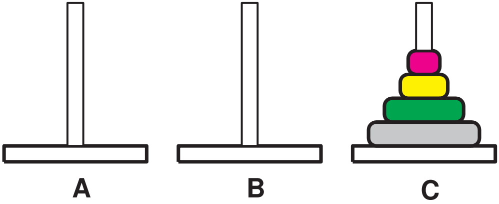
We can solve this problem by applying the following strategy:

First, we move the $(n-1)$ top discs of A on the B rod, and we use the empty rod C as an auxiliary one.

Then we move the largest disc of A on C rod.



In the last step, by applying the same algorithm we move the discs from B onto C and use A as an auxiliary rod.



Recursive algorithm to solve the problem of Hanoi towers

```
HanoiTowers (n, A, B, C)
begin
  (1)  if  (  n > 0  )
  (2)      HanoiTowers (n-1, A, C, B);
  (3)      move (A, C);
  (4)      HanoiTowers (n-1, B, A, C);
  (5)  end if
end  HanoiTowers
```

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Smaller subproblems again can be solved by applying the divide-and-conquere algorithm.

The recursive division is done until the received tasks are easily solved.