Sorting algorithms

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Main Problems

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There are quite many sorting algorithms and in applications it is important to select the one best fitted for the specific problem.

Problem formulation

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The output $A^{'} = (a_1^{'}, a_2^{'}, \ldots, a_N^{'})$ of any sorting algorithm must satisfy two conditions:

1. The output is in monotonic order (each element is no smaller/larger than the previous element, according to the required order)

$$a_i^{'} \leq a_{i+1}^{'}, \quad \text{if} \quad i = 0, 1, \ldots, N-1.$$

2. The output is a permutation (a reordering, yet retaining all of the original elements) of the input.

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Such algorithms can be presented as binary tree data structure.

Each vertex denotes a comparison operation of two elements, and its leaves define a result of this comparison.

Let us consider a simple example, when we need to sort three elements $A = (a, b, c)$.

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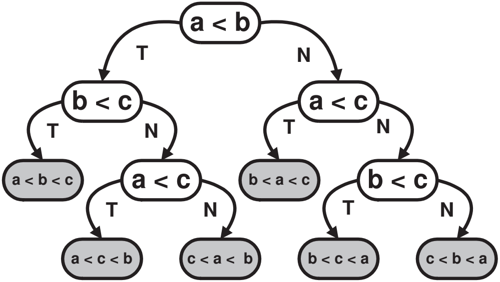
The sorting process and all possible sorted sets are presented in the figure.



T denotes an edge, when the condition is "true" and N otherwise.

It follows from the presented figure that three comparisons are required in the worst case.

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A more important conclusion is that the number of leaves of binary tree should be not smaller than a total number of possible permutations.

A full binary tree of height h has 2^h leaves.

For N elements it is possible to construct $N!$ different permutations, thus for any sorting algorithm a number of comparisons K must be not smaller than a solution of the inequality

$$2^K \geq N!$$

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In the case of three elements we get $K = 3$ since

$$2^3 \geq 3!$$

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Thus for any comparison based sorting algorithm the worst case complexity is estimated as

$$W_b \geq N \log N.$$

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This result is very unexpected, since the obtained complexity estimate O(N) is better than the lower estimate which was proved above for ANY sorting algorithm.

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In fact no mistakes were done in our analysis.

We must add costs of constructing an initial balanced search tree. As it was shown in previous lectures the costs of best algorithms are equal to $O(N \log N)$.

Next we consider two simple sorting algorithms. They are very useful when a small number of elements is sorted.

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Note, that the complexity analysis of these algorithms is also simple and it helps us to formulate main steps of such a task very clearly.

Selection sort

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The Selection sort divides the input list into two parts:

a sorted sublist of items which is built up from left to right at the front (left) of the list

and a sublist of the remaining unsorted items that occupy the rest of the list.

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The algorithm proceeds by finding the smallest (or largest) element in the unsorted sublist $i, i + 1, \ldots, N$,

exchange (swap) it with the leftmost unsorted element (putting it in sorted order), if required,

and move the sublist boundaries one element to the right, $i = i + 1$.

Selection sort algorithm

```
SelectionSort ()
begin
  (1)  for ( i = 1;  i < N ;  i++ ) do
  (2)      k = i;
  (3)      for ( j = i+1;  j <= N;  j++ ) do
  (4)          if ( a_j < a_k )  k = j;
           end do
  (5)      if ( k > i )  swap ( a_i, a_k );
       end do
end SelectionSort
```

101	17	33	2	24

a)

2	17	33	101	24

b) $i = 1, \ k = 4$

2	**17**	33	101	24

c) $i = 2, \ k = 2$

2	**17**	**24**	101	33

d) $i = 3, \ k = 5$

2	**17**	**24**	**33**	**101**

e) $i = 4, \ k = 5$

Sorting of integer numbers: i is the iteration number , k is the index of the smallest number in an unsorted sublist, items a_i and a_k are swapped, if required

Complexity of the selection sort algorithm

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- comparison of two items a_i and a_j,
- swapping of a_i and a_j.

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swapping of a_i and a_j.

Let $L(N)$ denotes a total number of comparisons and $S(N)$ a number of swappings when a set A of the size N is sorted.

In the selection sort algorithm all comparisons are done at each iteration and for any distribution of elements, thus

$$L(N) = \sum_{i=1}^{N-1} (N-i) = \sum_{i=1}^{N-1} i = \frac{N(N-1)}{2}.$$

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$$L(N) = \sum_{i=1}^{N-1} (N - i) = \sum_{i=1}^{N-1} i = \frac{N(N-1)}{2} \, .$$

This estimate is much worse than the bound $N \log N$.

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In the worst case such swappings are done at each iteration thus $S_B(N) = N - 1$.

Therefore the selection sort algorithm is recommended when a number of elements of A is not big, but the value field of each element is large.

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Efficient for data sets that are already substantially sorted: the time complexity is $O(kN)$, when each element in the input is no more than k places away from its sorted position;

It is stable, i.e. don't change the relative order of elements with equal keys.

Insertion sort iterates, taking one input element each repetition, and grows a sorted output list.

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Iterations are repeated until no input elements remain.

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At i-th iteration we take element a_i and insert it among already sorted $(i - 1)$ elements.

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The comparison is started from $(i-1)$-th element.

If $a_i < a_{i-1}$, then these elements are swapped and a next element a_{i-2} is tested.

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This process is continued till the correct place is defined.

Insertion sort algorithm

InsertionSort ()
begin
(1) **for** (i = 2; i <= N; i++) **do**
(2) v = a_i; a_0 =v; j = i;
(3) **while** ($v < a_{j-1}$) **do**
(4) $a_j = a_{j-1}$;
(5) j = j-1;
end do
(6) **if** (i $\neq$ j) $a_j = v$;
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A barrier technique is applied when a dummy element $a_0 = a_i$ is inserted before starting iterations (3).
This trick guarantee that iterations will end successfully without cheking if the first element is already reached.

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c)

2	**17**	**33**	**101**	24

d)

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e)

A grey color is used to denote the element which was inserted at a given iteration.

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If elements of A initially are distributed in an oposite order, then in step (1) all comparisons are done.

Thus in the *worst* case we get the complexity estimate

$$L_B(N) = \sum_{i=2}^{N} i = \frac{N^2 + N - 2}{2} = \frac{N^2}{2} + \mathcal{O}(N)\,.$$

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$$L_V(N) = \frac{N^2}{4} + \mathcal{O}(N)\,.$$

This estimate is only twice better than the worst case complexity.

Let us consider an interesting modification of this algorithm when comparison costs are much larger than swapping costs.

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Then we can apply the divide-and-conquere method in order to find the insertion place.

Let us assume that we want to insert element a_i.

First we compare a_i with the $a_{i/2}$.

If

$$a_{i/2} \leq a_i,$$

then we repeat this process in the interval $[i/2+1, i]$,

otherwise

we test the interval $[1, i/2-1]$.

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Assume that a_i must be inserted into j-th place.

Then we move elements a_j, ..., a_{i-1} to positions a_{j+1}, ..., a_i and insert the old a_i to the new position a_j.

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The total number of comparisons for this sorting algorithm is optimal

$$L_N = \mathcal{O}(N \log N).$$