Fast sorting algorithms

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In this lecture we consider three fast sorting algorithms. The complexity of them is close to the optimal estimate $O(N \log N)$.

Quicksort algorithm

Quicksort is an efficient, general-purpose sorting algorithm. It is still a very popular and commonly used in different applications algorithm.

We will show that its average complexity is $\mathcal{O}(N \log N)$, and Quicksort can be done in-place, requiring only small additional amounts of memory to perform the sorting.

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After applying this partition, Quicksort then recursively sorts the sub-sets.

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Elements that are equal to the pivot can go either way.

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A popular modification selects a small number M.

If the sub-set has fewer than M elements, sort it by some simple sorting algorithm, e.g. Insert sort.

Determination of the solution

Since no element of the first sub-set is greater than any element of the second sub-set, thus by sorting sub-sets we finish sorting all elements of A.

No computations are done at this stage.

Quicksort algorithm

```
QuickSort (l, r)
begin
  (1)  if  ( l < (r - M) )  then
  (2)       Partition ( l, r, m );
  (3)       QuickSort ( l, m-1 );
  (4)       QuickSort ( m+1, r );
       else
  (5)       if ( l < r ) SelectionSort (l, r);
       end if
end QuickSort
```

Partition (l, r, m)

begin

(1) $v = a_l$;

(2) i = l; j = r;

(3) **while** ($i < j$) **do**

(4) **while** (($a_j \geqslant v$) && ($i < j$)) $j = j - 1$;

(5) **if** ($i \neq j$) **then**

(6) $a_i = a_j$; i++;

end if

(7) **while** (($a_i \leqslant v$) && ($i < j$)) $i = i + 1$;

(8) **if** ($i \neq j$) **then**

(9) $a_j = a_i$; j−−;

end if

end do

(10) $a_i = v$; m = i;

end Partition

Let's sort a list

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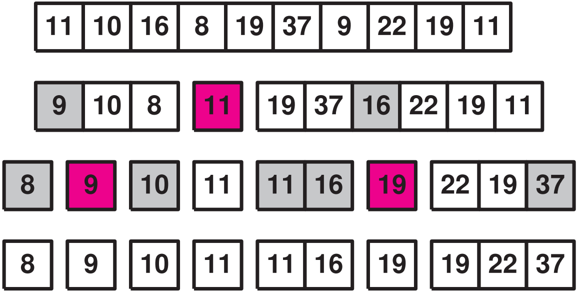

The first element of any sub-set is used as a pivot.
Pivots are colored red, grey colored elements are swaped during partition steps.

Complexity of Quicksort algorithm

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During a partition step each element is compared with a pivot.

Thus a total number of comparisons depends only on sizes of produced sub-sets.

Let's consider the worst case, when the smallest element is selected as a pivot.

Then we get the following equation

$$L_B(N) = L_B(N-1) + N - 1 \, .$$

If a set contains only one element then it is already sorted:

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Thus in the worst case this algorithm is not faster than Insert sort or Select sort algorithms.

The most un-expected conclusion is that such a result follows for already sorted sets (when the first element is selected as a pivot).

Let's consider the best case, when at each partition step we select the pivot element which divides a set into two sub-sets of equal sizes.

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Take $N = (2^m - 1)$. Then the number of comparisons satisfy the relation:

$$L_G(2^m - 1) = \begin{cases} 2L_G(2^{m-1} - 1) + 2^m - 2, & \text{when} \quad m > 1, \\ 0, & \text{when} \quad m = 1\,. \end{cases}$$

Applying it $(m-2)$ times we get

$$\begin{aligned} L_G(N) &= 2^m - 2 + 2\cdot(2^{m-1}-2) + 2^2\cdot(2^{m-2}-2) + \dots \\ &\quad + 2^{m-2}\cdot(2^2-2) \\ &= (m-1)2^m + 2^m - 2 \\ &= (N+1)\log(N+1) - 2. \end{aligned}$$

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But only for the best case.

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Thus the following two modfications of the base algorithm are recommended:

1. At each recursion stage three elements of A are selected in random a_k, a_l and a_m and they are sorted.

Then a mid element is taken as a pivot.

2. Before starting the Quicksort algorithm we swap all elements of A in random.

There is a big probability that sorting costs of such perturbed set will be close to the average complexity of Quicksort.

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In fact the task to find the median is a particular case of a more general second task

$$k = N/2.$$

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Now we will construct a fast algorithm by using the same divide-and-conquer method.

It is sufficient to modify a partition part of Quicksort algorithm.

Quick search algorithm

```
int QuickFind (l, r, k)    #  l ≤ k ≤ r
begin
 (1)  if  ( l == r )  then
 (2)      return (l);
      else
 (3)      Partition (l, r, m);
 (4)      if  ( m > k )  then
 (5)          QuickFind ( l, m-1, k );
          else
 (6)          if  ( m == k )  then
 (7)              return (m);
              else
 (8)              QuickFind ( m + 1, r, k );
      end if
end QuickFind
```

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Still at each stage only one recursion function is activated.

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It is recommended to present also an iterative version of this algorithm.

We restrict to the complexity analysis of the best case.

After each step of the partition algorithm the size of sub-sets is reduced twice, thus we get the following equation

$$L_G(N) = N + \frac{N}{2} + \frac{N}{4} + \ldots + 2 + 1 = 2N + \mathcal{O}(1)\,.$$

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Thus the median is computed $\frac{1}{2} \log N$ times faster than by using the Quicksort algorithm.

Merge sort

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It is interesting to note that divide-and-conquer method is again used to construct Merge sort.

Let's consider how three main steps of the divide-and-conquer method are implemented for this algorithm.

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Thus once again the recursion method is used.

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It is important to note that the Merge sort is a stable sort algorithm.

Let's apply the Merge sort for the following set of elements

$$A = (11,\ 10,\ 16,\ 8,\ 19,\ 37,\ 9,\ 22,\ 19,\ 11).$$

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8	19

37	9	22

19	11

10	11	16

8	19

9	22	37

11	19

8	10	11	16	19

9	11	19	22	37

8	9	10	11	11	16	19	19	22	37

Selection sort is used to sort grey color elements.

Complexity analysis of Merge sort

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For simplicity of analysis we take $N = 2^m$. Then we get the following equation

$$L(N) = \begin{cases} 2\,L\big(\frac{1}{2}N\big) + cN, & \text{if } \ N > 2, \\ 1, & \text{if } \ N = 2\,. \end{cases}$$

By applying this equation $(m-1)$ time, we get the total number of comparisons

$$\begin{aligned} L(N) &= 2L\left(\frac{1}{2}N\right) + cN \\ &= 4L\left(\frac{1}{4}N\right) + 2cN \\ &= \cdots = \frac{N}{2}L(2) + (m-1)cN \\ &= cN\log N + (0.5-c)N\,. \end{aligned}$$

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For large N and a randomly ordered input list, Merge sort's expected (average) number of comparisons of one step approaches αN fewer than the worst case, where $\alpha = 0.2645$.

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In the worst case, Merge sort uses approximately 39 percents fewer comparisons than Quicksort does in its average case.

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Next we formulate the main reasons why it is recommended to select QuickSort algorithm.

1. QuickSort implementations are faster than Merge sort.
2. QuickSort works in-place.

Heap sort algorithm

Let's recall main properties of heap data data structure.

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1. The value of each vertex is larger or equal to values of its children.

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4. The complexity of heap construction algorithm is $\Theta(N)$.

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3. The properties of the heap structure are restored if they where violated by the swap operation.

In order to make these modifications we use
HeapDownOrder (1, size(P)).

Heap sort algorithm

HeapSort ()
begin
(1) MakeHeap ();
(2) **for** (i=N; i > 1; i = i-1) **do**
(3) swap (a_1, a_i);
(4) HeapDownOrder (1, i-1);
end do
end HeapSort

10	37	18	13	22	14	25	8	12	28

a)

37	**28**	**25**	**13**	**22**	**14**	**18**	**8**	**12**	**10**

b)

10	28	25	13	22	14	18	8	12	**37**

c)

28	22	25	13	10	14	18	8	12	**37**

d)

12	22	25	13	10	14	18	8	**28**	**37**

e)

25	22	18	13	10	14	12	8	**28**	**37**

f)

8	22	18	13	10	14	12	**25**	**28**	**37**

g)

22	13	18	8	10	14	12	**25**	**28**	**37**

h)

12	13	18	8	10	14	**22**	**25**	**28**	**37**

i)

18	13	14	8	10	12	**22**	**25**	**28**	**37**

j)