Lecture 3

Integration by substitution (Change of variables).

Let's calculate

$$\int_a^b f(x)\,dx$$

by changing $x = \varphi(t)$ and

1. f is continuous on $[a, b]$,
2. $\varphi(t)$ and $\varphi'(t)$ are continuous on $[\alpha, \beta]$,
3. $\varphi : [\alpha, \beta] \to [a, b]$ (the range of φ is $[a, b]$)
4. For lower and upper limits we have $\varphi(\alpha) = a, \quad \varphi(\beta) = b.$

Then

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt.$$

Proof. Take a primitive function $F(x)$:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$= F(\varphi(\beta)) - F(\varphi(\alpha))$$

$$= \int_\alpha^\beta d\,F(\varphi(t)) = \int_\alpha^\beta F'(\varphi(t))\,dt.$$

and

$$F'(\varphi(t)) = \frac{d}{dx} F(\varphi(t)) \cdot \varphi'(t) = f(\varphi(t))\varphi'(t).$$

Thus it is proved that

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt$$

Example. Calculate $\int_0^a \sqrt{a^2-x^2}\,dx$

We change variable x by

$$x = a\sin t, \quad 0 \le t \le \frac{\pi}{2}.$$

$$dx = a\cos t\,dt$$

$$\int_0^a \sqrt{a^2-x^2}\,dx = \int_0^{\pi/2} a^2\sqrt{1-\sin^2 t}\cos t\,dt$$

$$= a^2 \int_0^{\pi/2} \cos^2 t\,dt = \frac{\pi a^2}{4}$$

<u>Theorem 2</u>. If function is continuous on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f \text{ is even,} \\ 0, & \text{if } f \text{ is odd. function} \end{cases}$$

◬ Assume, that $\boxed{f(x) = -f(-x)}$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$t = -x$
$dt = -dx.$

$$= -\int_a^0 \underbrace{f(-t)}_{=-f(t)}\,dt + \int_0^a f(x)\,dx$$

$$= -\int_0^a f(t)\,dt + \int_0^a f(x)\,dx = 0.$$ ◮

Integration By Parts

<u>Theorem 3.</u> Assume that $u(x)$ and $v(x)$ are differentiable (continuously) on $[a, b]$. Integration by parts is given by the following formula:

$$\int_a^b u\,dv = u(x)v(x)\Big|_a^b - \int_a^b v\,du.$$

<u>Proof.</u>

$$d(uv) = u\,dv + v\,du$$

Then

$$\int_a^b d(uv) = \int_a^b u\,dv + \int_a^b v\,du$$

$$\shortparallel$$

$$u(x)V(x)\Big|_a^b$$

$\boxtimes$

<u>Example</u>. Calculate $\int_1^e x \ln x \, dx$

$$\int_1^e x \ln x \, dx = \int_1^e \ln x \, d\frac{x^2}{2}$$

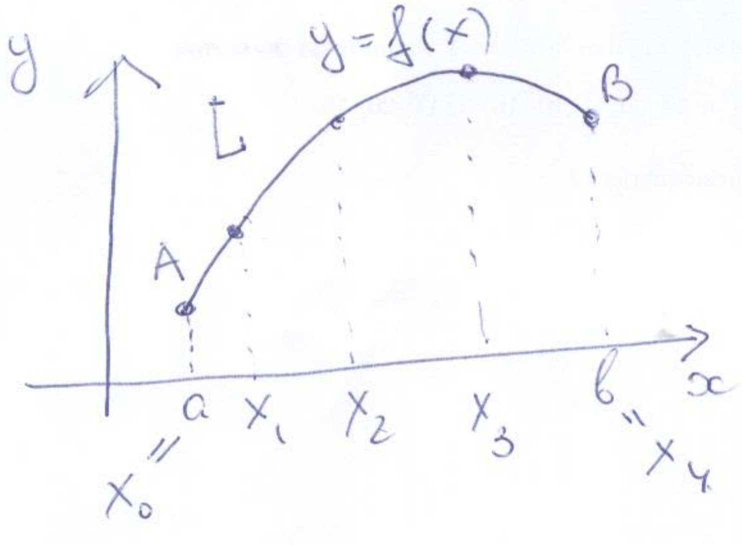
$$= \frac{x^2}{2} \ln x \Big|_1^e - \int_1^e \frac{x^2}{2} \, d \ln x$$

$$= \frac{e^2}{2} - \frac{1}{2} \int_1^e x \, dx = \frac{e^2+1}{4}$$

Applications. Geometry.

Using integration to find

Arc Lengths of a curve



<u>Definition</u>. The arc length of the curve $y = f(x)$, $a \le x \le b$ is equal to the limit of a sum of line approximations of this curve

$$L = \lim_{\Delta \to 0} \sum_{i=1}^{n} \Delta s_i$$

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i$$

Since $\dfrac{\Delta y_i}{\Delta x_i} = \dfrac{f'(c_i)\Delta x_i}{\Delta x_i} = f'(c_i)$

$$x_{i-1} \le c_i \le x_i$$

we get

$$\Delta s_i = \sqrt{1 + (f'(c_i))^2}\, \Delta x_i$$

and

$$\boxed{L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx}$$

<u>Example 1</u>. Find the arc length of the curve $y = x^{1.5}$, $0 \le x \le 2$.

$$y' = 1.5\sqrt{x}$$

$$L = \int_0^2 \sqrt{1 + \frac{9}{4}x}\, dx$$

$$= \frac{4}{9}\int_0^2 \sqrt{1 + \frac{9}{4}x}\; d\left(\frac{9}{4}x + 1\right)$$

$$= \frac{4}{9}\cdot\frac{2}{3}\left(1 + \frac{9}{4}x\right)^{3/2}\Bigg|_0^2 = \frac{8}{27}\left(\left(\frac{11}{2}\right)^{3/2} - 1\right)$$

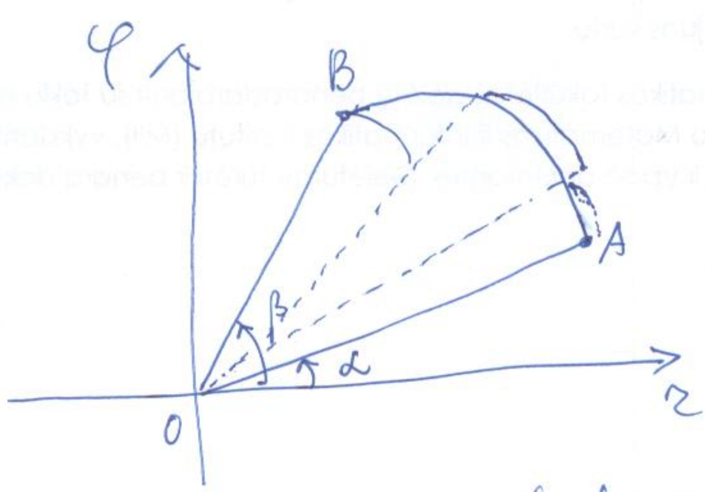
<u>Example 2</u>

Find the formula in the case when a curve is defined in parametric form

$$x = \varphi(t), \quad y = \psi(t), \quad t \in [t_0, T]$$

Area of regions bounded by polar curves

Now we define formula for calculation the area of a region given in polar coordinates.



Consider a curve defined by the function

$$r = f(\varphi), \quad \alpha \le \varphi \le \beta.$$

1. The first step is ~~to~~ partition the interval into subintervals

$$P = \{ \varphi_0 = \alpha < \varphi_1 < \dots < \varphi_n = \beta \}$$

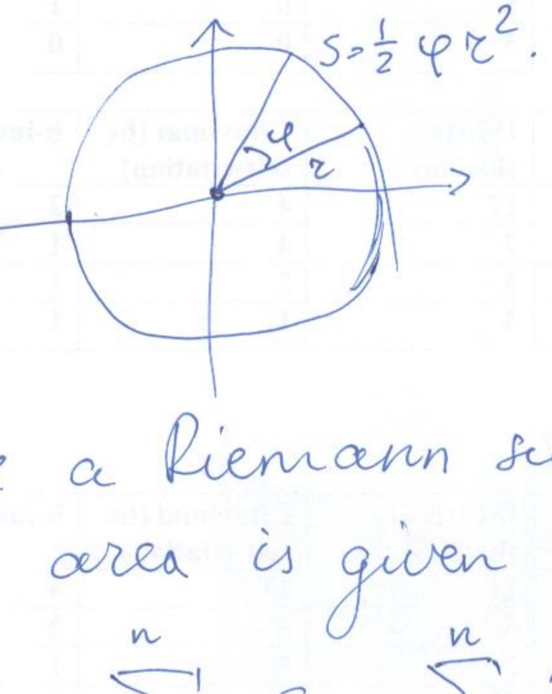
$$\Delta \varphi_i = \varphi_i - \varphi_{i-1}, \quad i = 1, 2, \dots, n.$$

2. The regions in each subintervals are replaced by sectors with radius

$$r_i = f(\varphi_i)$$

The ~~???~~ areas of these sectors are calculated as

$$\Delta S_i = \frac{1}{2} r_i^2 \Delta\varphi_i = \frac{1}{2} f^2(\varphi_i) \Delta\varphi_i$$



$$\left(S = \left(\frac{\varphi}{2\pi}\right) \pi r^2 = \frac{1}{2} r^2 \varphi \right)$$

Therefore a Riemann sum that approximates the full area is given by

$$S_n = \sum_{i=1}^{n} \Delta S_i = \sum_{i=1}^{n} \frac{1}{2} f^2(\varphi_i) \Delta\varphi_i$$

We take the limit as $\Delta\varphi \to 0$ and get the exact area

$$S = \lim_{\Delta\varphi \to 0} S_n = \frac{1}{2} \int_{\alpha}^{\beta} f^2(\varphi)\, d\varphi .$$

<u>Example.</u> Calculate areas bounded by curves:

$$\rho = ae^{0.2\varphi}, \quad 0 \le \varphi \le 2\pi$$

and

$$\rho = ae^{0.2\varphi}, \quad 2\pi \le \varphi \le 4\pi.$$

$$S = \frac{1}{2}\int_{2\pi}^{4\pi} a^2 e^{0.4\varphi}\,d\varphi - \frac{1}{2}\int_{0}^{2\pi} a^2 e^{0.4\varphi}\,d\varphi.$$

$$= \ldots\ldots = 1.25a^2\left(e^{0.8\pi}-1\right)^2.$$