Lecture 6

Triple integrals

A triple integral is a definite integral of a function of three variables over a region $\mathcal{D}$ in $\mathbb{R}^3$.

Let's assume that $f(x, y, z)$ is defined in $\mathcal{D} \subset \mathbb{R}^3$.

1. D is bounded

2. A surface $\partial \mathcal{D}$ is smooth

Example Let's assume that a mass density of a body P is defined by

$$\rho = \rho(x, y, z) \quad \left[\frac{kg}{m^3}\right].$$

Then the total mass of this body can be calculated as

$$M = \iiint_{D} \rho(x, y, z)\, dx\, dy\, dz.$$

Now let's give a definition of a <u>definite triple integrals</u>

1. Function $f = f(x, y, z)$ is continuous in $D \subset \mathbb{R}^3$.

2. Consider a partition of D into a finite number of non-overlapping sub-volumes

$$D = D_1 \cup D_2 \cup \dots \cup D_n.$$

3. We denote by ΔV_i the <u>volume</u> of D_i, $i = 1, \dots, n$.

d_i is the diameter of D_i

$d = \max\limits_{1 \le i \le n} d_i$ is the value of the largest diameter

4. Select P_i – a point in D_i (any point)

$$P_i = (x_i, y_i, z_i) \quad i = 1, \dots, n$$

5. Calculate the Riemann sum

$$\sigma_n = \sum_{i=1}^{n} f(P_i) \Delta V_i .$$

<u>Definition</u>. The function f is said to be Riemann integrable in region D if the limit

$$V = \lim_{d \to 0} \sum_{i=1}^{n} f(P_i) \Delta V_i$$

<u>exists</u> and <u>don't depend</u> on partition D_i and points P_i.

This integral is denoted

$$V = \iiint_D f(x,y,z)\,dx\,dy\,dz$$

$\left(\text{or } \iiint_D f(x,y,z)\,dV \right).$

If $f(x,y,z)$ is continuous then this integral exists.

This result can be proved by using Darboux sums.

Remark 1. If $f(x,y,z) \equiv 1$, then we get that the volume of (body) region D is calculated as

$$V = \iiint_D dx\,dy\,dz.$$

Methods of integration

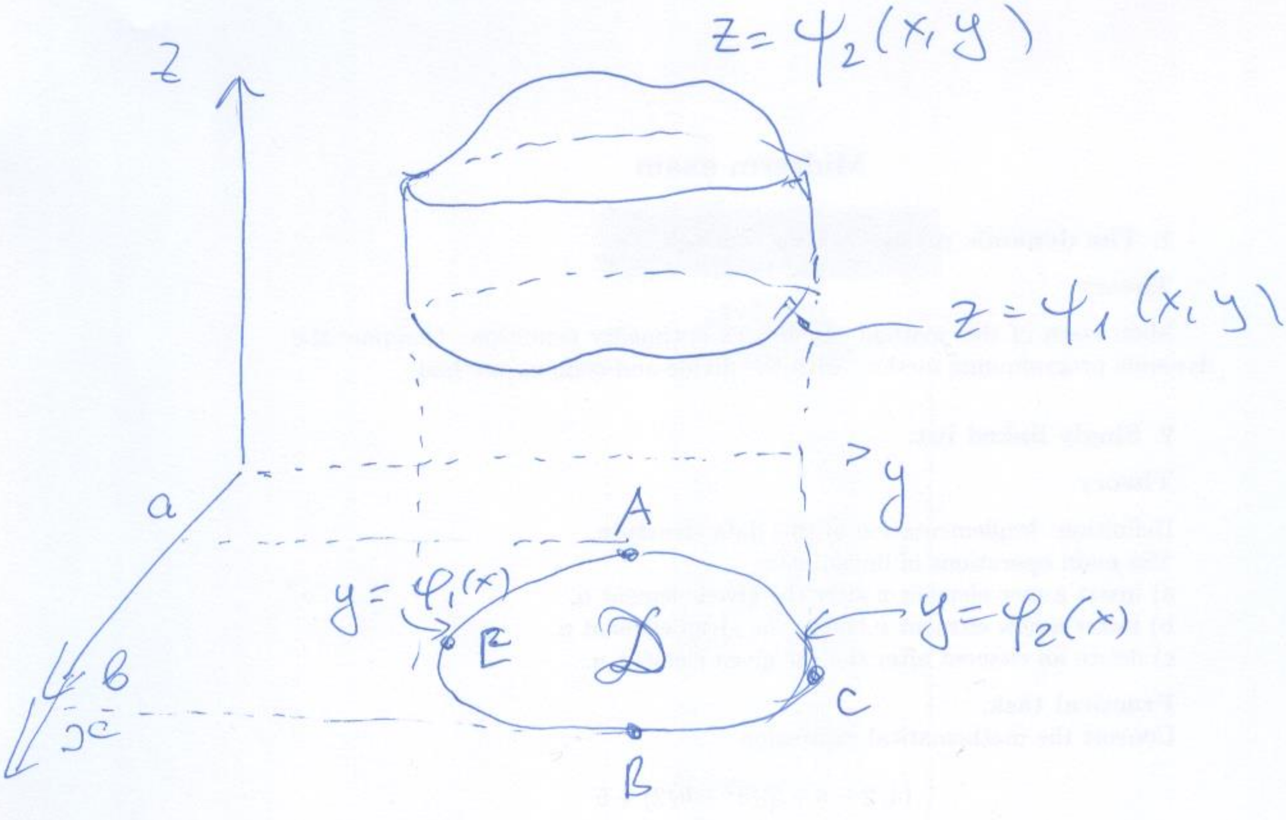
As for double integrals in the case of triple integrals the main idea is to reduce the triple integral to an iterated integral, i.e. three integrals of one integration variable.

Let's assume that the integration region V is defined as

$$a \le x \le b,$$

$$\varphi_1(x) \le y \le \varphi_2(x),$$

$$\psi_1(x,y) \le z \le \psi_2(x,y)$$



$\varphi_1(x)$, $\varphi_2(x)$ are continuous and unique in the interval $[a, b]$.

$\psi_1(x,y)$, $\psi_2(x,y)$ are continuous in $\mathcal{D}$.

The region V is bounded by $\psi_1(x,y)$ and from above by $\psi_2(x,y)$.

From sides it is bounded by cylinders.

The triple integral can be defined as an iterated integral

$$\iiint\limits_V f(x,y,z)\,dx\,dy\,dz$$

$$= \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dy \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz.$$

The proof is very similar ~~to~~ the analysis of double integrals.

Example 1.

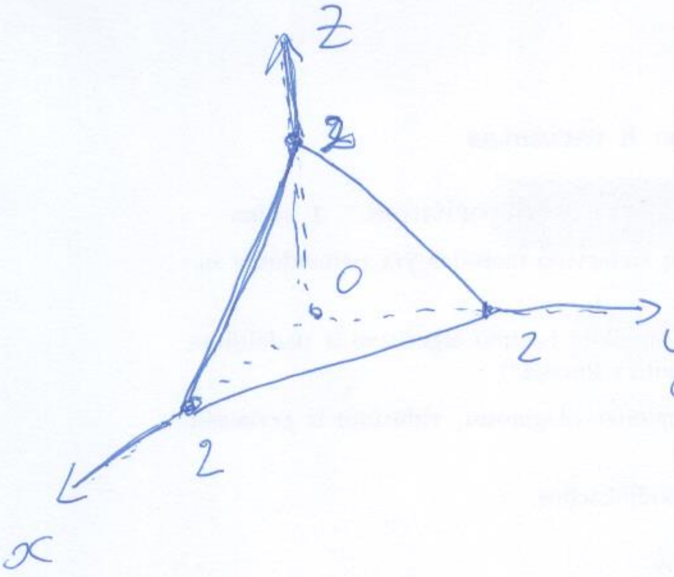
$$\iiint\limits_V (x+y-z)\,dx\,dy\,dz$$

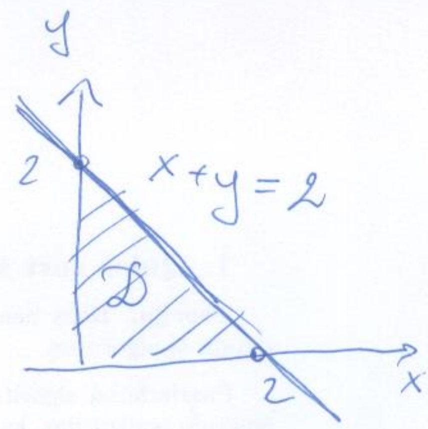
V: The region is bounded by planes

$x=0$

$y=0$ $\quad z=0.$

$x+y+z=2$





$$\iiint\limits_V (x+y-z)\,dx\,dy\,dz$$

$$= \int_0^2 dx \int_0^{2-x} dy \int_0^{2-x-y} (x+y-z)\,dz$$

$$(1) \quad = \int_0^2 dx \int_0^{2-x} \left(xz+yz-\frac{z^2}{2}\right)\Big|_0^{2-x-y} dy$$

$$= \int_0^2 dx \int_0^{2-x} \left(4x+4y-3xy-\frac{3}{2}x^2-\frac{3}{2}y^2-2\right) dy$$

$$(2) \quad = \int_0^2 \left(4xy+2y^2-\frac{3}{2}xy^2-\frac{3}{2}x^2y-\frac{1}{2}y^3-2y\right)\Big|_0^{2-x} dy$$

$$\dots\dots = \frac{2}{3}$$

Change of variables in Triple integrals

1) Let's suppose that regions V and $\tilde{V}$ are connected by the following relations

$$x = \varphi(u, v, w) \quad \text{phi } \varphi$$

$$y = \psi(u, v, w) \quad \text{psi } \psi$$

$$z = \eta(u, v, w) \quad \eta \text{ eta}$$

2) We assume that a surface of V (denoted by S) is connected to $\tilde{S}$, a surface of $\tilde{V}$.

We consider a triple integral

$$\iiint_V f(x, y, z)\, dx\, dy\, dz$$

Let's assume that functions φ, ψ and η are differentable

The Jacobian of transformation $I(u, v, w)$ is equal to

$$I(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

We assume that it is non-zero and keeps a constant sign in the region of integration V.

The formula for change of variables in triple integrals is written as

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{\tilde{V}} f(\varphi,\psi,\eta)\, I(u,v,w) \times du\,dv\,dz.$$

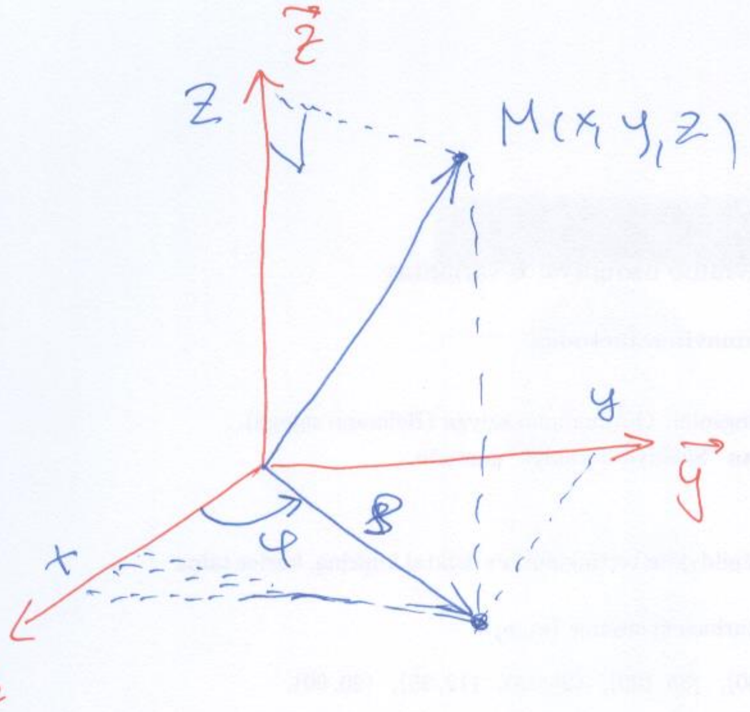
Triple Integrals in Cylindrical Coordinates

The position of a point $M(x, y, z)$ in the x, y, z - coordinates (space) Cartesian coordinates in cylindrical coordinates is defined by three numbers: ρ, φ, z, where

ρ - is the projection of the radius vector of the point M onto the xy-plane

φ - is the angle formed by the projection of the radius vector with x-axis

z - is the projection of the radius vector on the z-axis (the same value in Cartesian and cylindrical coordinates.)



The relationship between cylindrical and Cartesian coordinates of a point M is given by

$$x = \rho \cos\varphi, \quad y = \rho \sin\varphi, \quad z = z$$

$$\rho \geq 0, \quad 0 \leq \varphi \leq 2\pi, \quad -\infty < z < \infty.$$

$$J(\rho,\varphi,z) = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho \geq 0$$

$$\iiint\limits_V f(x,y,z) = \iiint\limits_{\tilde V} f(\rho\cos\varphi, \rho\sin\varphi, z)\,\rho\, d\rho\, d\varphi\, dz$$

Example 1. Evaluate the integral

$$\iiint\limits_V (x^4 + 2x^2y^2 + y^4)\,dx\,dy\,dz$$

where V is bounded by the surface

$x^2 + y^2 \leq 1$ and the planes $z=0$, $z=1$

$$x^4 + 2x^2y^2 + y^4 = (x^2+y^2)^2 = \rho^4.$$

Then the integral becomes

$$I = \int_0^{2\pi} d\varphi \int_0^1 \rho^5\, d\rho \int_0^1 dz = \frac{\pi}{3}.$$

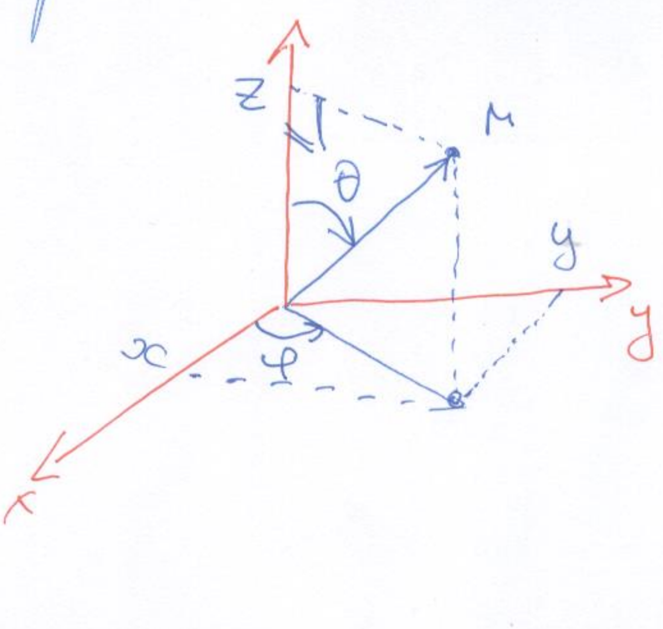
Spherical Coordinates

The spherical coordinates of a point $M(x, y, z)$ are defined by the three numbers

ρ is the length of the radius vector to the point M

φ is the angle between the projection of the radius vector $\overrightarrow{OM}$ on the xy-plane and x-axis

θ is the angle of vector $\overrightarrow{OM}$ from the the positive direction of the z-axis



The spherical coordinates of M are related to its Cartesian coordinates as follows

$$x = \rho\cos\varphi\sin\theta, \quad y = \rho\sin\varphi\sin\theta,$$

$$z = \rho\cos\theta$$

$$\rho \geq 0, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \leq \theta \leq \pi.$$

The Jacobian of transformation from Cartesian to spherical coordinates is written as

$$I(\rho, \varphi, \theta) = \begin{vmatrix} \cos\varphi\sin\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\theta & 0 & -\rho\sin\theta \end{vmatrix}$$

$$= \rho^2\sin\theta.$$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{\tilde{V}} f(\cancel{\varphi, \psi, \eta}\ \rho\cos\varphi\sin\theta, \rho\sin\varphi\sin\theta, \rho\cos\theta)\,\rho^2\sin\theta\,d\rho\,d\varphi\,d\theta.$$

<u>Example</u> Calculate the integral

$$\iiint_V e^{(x^2+y^2+z^2)^{3/2}} \, dx\, dy\, dz$$

where the region V is the unit ball

$$x^2+y^2+z^2 \le 1$$

In spherical coordinates the region V is described by the inequalities

$$0 \le \rho \le 1, \quad 0 \le \varphi \le 2\pi, \quad 0 \le \theta \le \pi$$

$$\iiint_V e^{(x^2+y^2+z^2)^{3/2}} \, dx\, dy\, dz$$

$$= \iiint_{\tilde V} e^{(\rho^2)^{3/2}} \rho^2 \sin\theta \, d\rho\, d\varphi\, d\theta$$

$$= \iiint_{\tilde V} e^{\rho^3} \rho^2 \sin\theta \, d\rho\, d\varphi\, d\theta$$

$$= \frac{4\pi}{3}(e-1)$$