Fast sorting algorithms

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In this lecture we consider three fast sorting algorithms. The complexity of them is close to the optimal estimate $O(N \log N)$.

Quicksort algorithm

Quicksort is an efficient, general-purpose sorting algorithm. It is still a very popular and commonly used in different applications algorithm.

We will show that its average complexity is $\mathcal{O}(N \log N)$, and Quicksort can be done in-place, requiring only small additional amounts of memory to perform the sorting.

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After applying this partition, Quicksort then recursively sorts the sub-sets.

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Elements that are equal to the pivot can go either way.

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A popular modification selects a small number M.

If the sub-set has fewer than M elements, sort it by some simple sorting algorithm, e.g. Insert sort.

Determination of the solution

Since no element of the first sub-set is greater than any element of the second sub-set, thus by sorting sub-sets we finish sorting all elements of A.

No computations are done at this stage.

Quicksort algorithm

QuickSort (l, r)
begin
(1) **if** (l < (r - M)) **then**
(2) Partition (l, r, m);
(3) QuickSort (l, m-1);
(4) QuickSort (m+1, r);
else
(5) **if** (l < r) SelectionSort (l, r);
end if
end QuickSort

Partition (l, r, m)
begin

(1) $v = a_l$;
(2) $i = l$; $j = r$;
(3) **while** $(i < j)$ **do**
(4)   **while** $((a_j \geqslant v) \; \&\& \; (i < j))$ $j = j - 1$;
(5)   **if** $(i \neq j)$ **then**
(6) $a_i = a_j$; i++;
    end if
(7)   **while** $((a_i \leqslant v) \; \&\& \; (i < j))$ $i = i + 1$;
(8)   **if** $(i \neq j)$ **then**
(9) $a_j = a_i$; j−−;
    end if
end do
(10) $a_i = v$; $m = i$;

end Partition

Let's sort a list

$$A = (11, 10, 16, 8, 19, 37, 9, 22, 19, 11).$$

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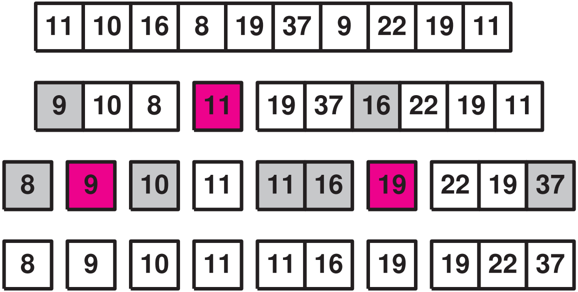

The first element of any sub-set is used as a pivot.
Pivots are colored red, grey colored elements are swaped during partition steps.

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During a partition step each element is compared with a pivot.

Thus a total number of comparisons depends only on sizes of produced sub-sets.

Let's consider the worst case, when the smallest element is selected as a pivot.

Then we get the following equation

$$L_B(N) = L_B(N-1) + N - 1\,.$$

If a set contains only one element then it is already sorted:

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Thus in the worst case this algorithm is not faster than Insert sort or Select sort algorithms.

The most un-expected conclusion is that such a result follows for already sorted sets (when the first element is selected as a pivot).

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Take $N = (2^m - 1)$. Then the number of comparisons satisfy the relation:

$$L_G(2^m - 1) = \begin{cases} 2L_G(2^{m-1} - 1) + 2^m - 2, & \text{when} \quad m > 1, \\ 0, & \text{when} \quad m = 1\,. \end{cases}$$

Applying it $(m-2)$ times we get

$$\begin{aligned} L_G(N) &= 2^m - 2 + 2 \cdot (2^{m-1} - 2) + 2^2 \cdot (2^{m-2} - 2) + \dots \\ &\quad + 2^{m-2} \cdot (2^2 - 2) \\ &= (m-1)2^m + 2^m - 2 \\ &= (N+1)\log(N+1) - 2. \end{aligned}$$

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But only for the best case.

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1. At each recursion stage three elements of A are selected in random a_k, a_l and a_m and they are sorted.

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2. Before starting the Quicksort algorithm we swap all elements of A in random.

There is a big probability that sorting costs of such perturbed set will be close to the average complexity of Quicksort.