Lecture 10

The Laplace transform

This transform defines the very efficient method ~~to~~ solve some classes of ODEs.

It can transform ODE ~~to~~ an algebraic equation. The largest are of applications arise from electrical circuits.

Def. Let us define the Laplace transform

$$\mathcal{L}\{f(t)\} = F(s) \overset{def}{:=} \int_0^\infty e^{-st} f(t)\,dt.$$

$t > 0$, $t \in \mathbb{R}$ $f(t)$ is a signal

$s \in \mathbb{C}$ a complex number

F is a complex-valued function of complex numbers

s is the frequency variable (complex), with units $\sec^{-1}$,

t – time variable in sec,

We assume that $f(t)$ contains no impulses at $t=0$.

A common notation convention:

$$U = \mathcal{L}(u), \quad V = \mathcal{L}(v).$$

<u>Example 1.</u>

$$f(t) = e^{t}, \quad t > 0.$$

$$f(t) \equiv 0, \quad t \le 0.$$

$$F(s) = \int_0^\infty e^{t} e^{-st}\, dt = \int_0^\infty e^{(1-s)t}\, dt$$

$$= \frac{1}{1-s} e^{(1-s)t} \Big|_0^\infty = \frac{1}{s-1}$$

provided that $e^{(1-s)t} \to 0$ as $t \to \infty$,

which is true for Real $s > 1$,

$$\Re s > 1$$

$$|e^{(1-s)t}| = |e^{-i(\Im s)t}||e^{(1-\Re s)t}|$$

$$\overset{\shortparallel}{1}$$

$$= e^{(1-\Re s)t}.$$

The region of convergence of F

$s \in \mathbb{C}$ with $\Re s > 1$

$$\mathcal{L}(e^t) = \frac{1}{s-1}.$$

<u>Example 2</u> $f(t) = 1$ for $t \geq 0$

$$F(s) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$$

provided that $e^{-st} \to 0$ as $t \to \infty$,

which is true for $\Re s > 0$ since

$$|e^{-st}| = \underbrace{|e^{-i(\Im s)t}|}_{=1} \, \underbrace{|e^{-(\Re s)t}|}_{= e^{-(\Re s)t}}$$

- the integral defining $F(s)$ makes sense for all s with $\Re s > 0$ but the resulting formula for F makes sense for all s except $s = 0$.

Example 3 $\quad f(t) = \cos \omega t$

First we express $f(t)$ as

$$f(t) = \frac{1}{2} e^{i\omega t} + \frac{1}{2} e^{-i\omega t}$$

Now we find F as

$$F(s) = \int_0^\infty e^{-st} \left(\frac{1}{2} e^{i\omega t} + \frac{1}{2} e^{-i\omega t} \right) dt$$

$$= \frac{1}{2} \int_0^\infty e^{(-s+i\omega)t} dt + \frac{1}{2} \int_0^\infty e^{(-s-i\omega)t} dt$$

$$= \frac{1}{2} \left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega} \right) = \frac{s}{s^2+\omega^2}$$

Formula is valid for $\Re s > 0$

Example 4 $\quad f(t) = t^n, \; n \geq 1.$

We integrate by parts:

$$F(s) = \int_0^\infty t^n e^{-st}\, dt$$

$$= t^n \left(\frac{-e^{-st}}{s} \right) \Bigg|_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st}\, dt$$

$$= \frac{n}{s} \mathcal{L}\left(t^{n-1}\right)$$

provided $t^n e^{-st} \to 0$ if $t \to \infty$, which is true for $\Re s > 0$

Applying the formula recursively we obtain

$$F(s) = \frac{n!}{s^{n+1}}.$$

Example $f(t) = \delta(t)$.

$$F(s) = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = e^{-st}\Big|_{t=0} = 1.$$

Linearity:

The Laplace transform is linear: f and g are signals and a is scalar, we have

$$\mathcal{L}(af) = aF,$$

$$\mathcal{L}(f+g) = F + G.$$

Example.

$$\mathcal{L}(5\delta(t) - 7e^{t})$$

$$= 5\mathcal{L}(\delta(t)) - 7\mathcal{L}(e^{t})$$

$$= 5 - \frac{7}{s-1} = \frac{5s-12}{s-1}$$

Inverse Laplace transform

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}\,ds$$

where σ is large enough that $F(s)$ is defined for $\Re s > \sigma$.