Lecture 1

Complex numbers

Real numbers $\mathbb{R}$ are defined and used in the mathematical analysis for many (most) topics.

Still we need some generalization of real numbers in order to have a more efficient tool.

We discuss a new system of numbers that contains the real numbers and permits the solution of the equation

$$\boxed{x^2 = -1}$$

Definition 1 A complex number is an ordered pair (x, y) of real numbers $x, y \in \mathbb{R}$. Addition and multiplication of complex numbers are defined by:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$$

Example

$$(2, 5)(3, 7) = (2 \cdot 3 - 5 \cdot 7, 2 \cdot 7 + 3 \cdot 5)$$

$$= (-29, 29)$$

Theorem 1 The following axioms hold

if $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$ are complex number, then

(a) Addition and multiplication are commutative

$$z_1 + z_2 = z_2 + z_1$$

$$z_1 z_2 = z_2 z_1$$

(b) Addition and multiplication are associative:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

(c) Additive identity $(0, 0) := 0$

$$z_1 + 0 = z_1$$

(d) Multiplicative identity $(1, 0) := 1$

$$z_1 \cdot 1 = z_1$$

(e) Each element has an additive inverse:

$$(x, y) + (-x, -y) = (0, 0).$$

(f) Each element other than $(0,0)$ has a multiplicative inverse:

$$(x, y)\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = (1, 0)$$

(g) Multiplication distributes over addition

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

Proof. $(x, y) = x + iy$, $i^2 = -1$

Proof of (f). $(x+iy)(a+ib) = 1$

$$(x-iy)(x+iy)(a+ib) = x - iy$$

$$\underbrace{(x^2+y^2)}_{\in \mathbb{R}}(a+ib) = x - iy$$

$$a = \frac{x}{x^2+y^2}, \quad b = -\frac{y}{x^2+y^2}$$

$\triangleright$

The $x + iy$ notation

We define the complex number

$$i = (0, 1)$$

Then (x, y are real numbers)

The Cartesian form

$$x + iy \qquad x = (x, 0) \qquad y = (y, 0)$$

$$= (x, 0) + (0, 1)(y, 0) = (x, y)$$

$$i^2 = (0,1)(0,1) = (0.0 - 1.1,\ 0.1 + 1.0)$$

$$= (-1, 0) = -1.$$

$$\Rightarrow \quad i = \sqrt{-1}.$$

Prz $\quad 1 + i2 = \underbrace{1 + 2i}_{}$

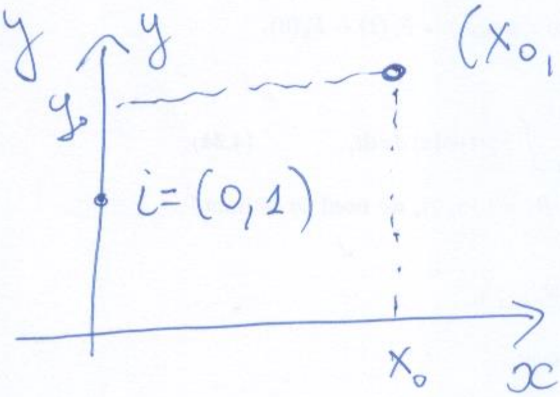
We will use this form as the main form.

<u>Example</u>.

$$\frac{2+3i}{1+2i} = \frac{2+3i}{1+2i}\,\frac{1-2i}{1-2i} = \frac{8-i}{5}$$

$$= \frac{8}{5} - \frac{i}{5}.$$

$x+iy$



$(x_0, y_0) = x_0 + iy_0$.

$z = x+iy$

<u>x</u> is the <u>real</u> part of z. $\boxed{x = \mathrm{Re}(z)}$

<u>y</u> is the imaginary part of z $\boxed{y = \mathrm{Im}(z)}$

$|z| = \sqrt{x^2+y^2}$ is the modulus, or <u>absolute value</u> of z.

If $z = x + iy$, its complex conjugate is

$$\bar{z} = x - iy.$$

The following properties are valid:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$$

$$re(z) = \frac{1}{2}(z + \bar{z})$$

$$im(z) = \frac{1}{2i}(z - \bar{z})$$

$$|z|^2 = z \cdot \bar{z}$$

The modulus has the following properties

1) $|z_1 + z_2| \le |z_1| + |z_2|$

(the triangle inequality)

2) $|z_1 z_2| = |z_1|\,|z_2|$

3) $\big|\, |z_1| - |z_2| \,\big| \le |z_1 - z_2|$.

Practical task Show that property (3) is a consequence of (1)

Polar Coordinates

The Cartesian and Polar coordinates are related as follows:

$x = r \cos\theta, \quad y = r \sin\theta.$

$r = |z| = \sqrt{x^2 + y^2}$

A value of θ is unique only up to multiples of 2π:

$$\theta = \arg z.$$

Any value of θ for which

$$x = r\cos\theta, \quad y = r\sin\theta$$

holds is called $\boxed{\text{an argument of } z}$

The value $\theta \in (-\pi, \pi]$ is called the <u>principal value</u> of the argument

Exercises

1 Find all solections $z = a + ib$ of the following equations:

1.1. $z^2 = -5 + 12i$

1.2. $z^2 = 2 + i$

1.3 $z^2 - 3z + 1 + i = 0$

2. Draw the set of points

$$\{ z \in \mathbb{C} : \mathrm{re}(z+1) = |z-1|$$

by substituting $z = x + iy$ and computing the real equation relating x and y

Draw the sets: $|z-1| < 1$.

$$z\bar{z} = 1.$$

3. Derive De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all natural numbers $n \in \mathbb{N}$.

Use the <u>induction</u> method

4. Define a square root $\sqrt{z}$ to be any complex number w such that

$$w^2 = z.$$

Give formulas for two square roots

5. If $a, b, c \in \mathbb{C}$ with $a \neq 0$ show that the solutions of the quadratic equation

$$az^2 + bz + c = 0$$

are:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$