Lecture 2

Functions

Now we collect the basic ideas required to study complex analysis

a) differentiation is set against of limits and continuity

b) we restrict to complex functions defined on open path connected sets, called domains.

Definition. For a complex number z_0 and positive real number ε, the ε-neighbourhood of z_0 is

$$N_\varepsilon(z_0) = \{ z \in \mathbb{C} : |z - z_0| < \varepsilon \}$$

This is an open disc of radius ε, A subset $S \subseteq \mathbb{C}$ is said to be <u>open</u> if for every $z_0 \in S$ there is a <u>real</u> number $\varepsilon \in \mathbb{R}$ such that

$$N_\varepsilon(z_0) \subseteq S.$$

Note, that ε may depend on z_0.

<u>Example</u>. The disc $N_\varepsilon(z_0)$ is itself open.

◁ If $z_1 \in N_\varepsilon(z_0)$ then $|z_1 - z_0| < \varepsilon$.

Choose $\delta > 0$ such that

$$\delta < \varepsilon - |z_1 - z_0|.$$

By the triangle inequality

$$z \in N_\delta(z_1) \Rightarrow |z - z_1| < \delta$$

$$|z - z_0| \leq |z - z_1| + |z_1 - z_0| \leq \delta + |z_1 - z_0| < \varepsilon$$

$$\Rightarrow N_\delta(z_1) \subseteq N_\varepsilon(z_0)$$ ▷

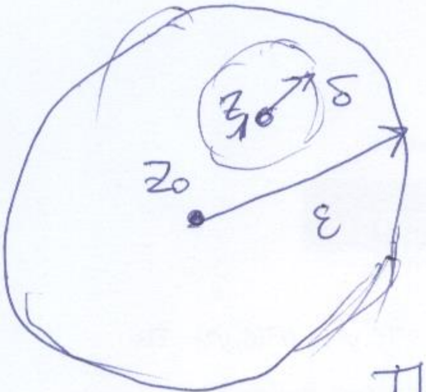

Fig. 1

The ε-neighbourhood of point z_0, and δ-neigh. of point z_1.

Definition. The complement of a subset $S \subseteq \mathbb{C}$ is

$$\mathbb{C} \setminus S = \{ z \in \mathbb{C} : z \notin S \}$$

A subset $\cancel{S}$ is a closed set if $\mathbb{C} \setminus S$ is open set.

Closed sets can be defined using the notion of a limit point of S.

$z_0 \in \mathbb{C}$ is a limit point of S if $\forall\, N_\varepsilon(z_0)$ contains a point $z_\varepsilon \in S$ not equal to z_0.

Proposition. A subset $S \subseteq \mathbb{C}$ is closed if and only if ($\Leftrightarrow$) S contains all its limit points.

Note. Not every point of a closed set needs to be a limit point of that set. E.g.:

$$T = \{z \in \mathbb{C} : z = 0 \text{ or } z = \frac{1}{n}, \text{ for } n \in \mathbb{N}\}$$

Then the only limit point of T is 0. Since 0 is in T, the set T is closed

Definition. A complex function

$$f : \mathcal{D} \subseteq \mathbb{C} \to \mathbb{C}$$

is a function from complex numbers to complex numbers: $\mathcal{D}$ is a domain, and complex numbers $f(z)$ a Codomain

$$f = f(z), \quad z \in \mathbb{C}.$$

If only one value of $w = f(z)$ corresponds to each value z, we say that w is a single-valued function. Ex.: $w = z^2$.

If more than one value of w corresponds to each value of z, we say that w is a multiple-valued function.

Ex.: $w = z^{1/2}$.

Notion of a limit

$$\lim_{z \to z_0} f(z)$$

Def. If $f : S \to \mathbb{C}$ is a complex function and z_0 is a limit point of S, then $\boxed{\lim_{z \to z_0} f(z) = \ell}$ if given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall\, z \in S, \quad 0 < |z - z_0| < \delta \Rightarrow$$

$$|f(z) - \ell| < \varepsilon.$$

(a) The point z_0 need not belong to S, so $f(z_0)$ need not be defined. !!

Ex.

$$f(z) = \begin{cases} 0, & \text{for } z \neq 0 \\ 1, & \text{for } z = 0 \end{cases}$$

Then

$$\lim_{z \to 0} f(z) = 0 \neq f(0).$$

If z_0 is a limit point of S and $\lim_{z \to z_0} f(z) = l$, then the limit is unique.

~~Make a proof of this result~~

$$\lim_{z \to z_0} f(z) = l, \quad \lim_{z \to z_0} g(z) = k$$

$$\lim_{z \to z_0} \left(f(z) \pm g(z) \right) = l \pm k.$$

$$\lim_{z \to z_0} \left(f(z) g(z) \right) = l \cdot k.$$

Continuity

<u>Def.</u> A function $f : S \to \mathbb{C}$ is continuous at $z_0 \in S$ if given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$\forall z \in S$, $|z - z_0| < \delta$ implies

$|f(z) - f(z_0)| < \varepsilon$.

A function f is continuous if it is continuous at every point $z_0 \in S$.

If z_0 is a limit point of S, this is equivalent to say

$$\exists \lim_{z \to z_0} f(z) \iff \lim_{z \to 0} f(z) = f(z_0)$$

$$\Big(= f\big(\lim_{z \to z_0} z \big)$$

<u>Example</u>. Let consider an isolated point z_0 of S. Then there is a neighbourhood ~~the~~ $N_\delta(z_0)$ that contains no other points of S apart from z_0:

$$\text{for all } z \in S,\ |z - z_0| < \delta \Rightarrow z = z_0$$

which implies

$$|f(z) - f(z_0)| = 0$$

So a complex function is always continuous at an isolated point

<u>Corollary</u>. If $S \subseteq \mathbb{C}$ is open, then

$f : S \to \mathbb{C}$ is continuous

$\iff$ for every open set U, the inverse image $f^{-1}(U)$ is open.

<u>Proposition.</u> If f is continuous at $z_0 \in S$ and g is continuous at $f(z_0)$ then composition

$$g \circ f : S \to \mathbb{C}$$

is continuous at z_0.

◁ Give a proof. ▷

<u>Proposition</u> If f_1 and f_2 are continuous, then so are functions

$$f_1 + f_2, \; f_1 - f_2, \; f_1 f_2, \; f_1 / f_2$$

(when $f_2(z) \neq 0$).

Exercises

1. Draw the sets, in each case state whether the set is open, closed or neither:

(i) $1 < |z| < 2$, (ii) $\operatorname{re} z \geq 0$

$1 < |z| < 2$.

(iii) $\operatorname{re}(z z_0) > 0$

(iv) $|z - \bar{z}_0| = |\bar{z} - z_0|$

2. Find the limits, if they exist

(i) $\lim\limits_{z \to 0} |z| / z$

(ii) $\lim\limits_{z \to 0} |z|^2 / z$, $\lim\limits_{z \to 0} z / |z|^2$

3. Prove that the following function are continuous

$\operatorname{re} z$, $\operatorname{im} z$, $|z| + z$, $|z|^2$