Lecture 3

Differentiation

We will get only one surprise (even minor) that the condition of differentiability of complex functions implies certain relations between the real and imaginary parts of complex functions (the Cauchy-Riemann equations).

The dramatic difference between real and complex theories of differentiation is the following: every differentiable complex function can be differentiated arbitrarily many times.

Definition.

(a) A complex function f defined on an open set S is differentiable at $z_0 \in S$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \underline{\text{exists}}.$$

(b) The value of this limit is then defined to be the derivative $f'(z_0)$ at that point.

(c) If f is differentiable at $\forall z_0 \in S$ then f is differentiable

$$f' : S \to \mathbb{C}.$$

(d) If f' is also differentiable, we define the second derivative

$$f''(z) = \lim_{z \to z_0} \frac{f'(z) - f'(z_0)}{z - z_0}$$

(Similarly we can define $f^{(n)}(z)$ at $z_0 \in S$.

<u>Example</u>. $f(z) = z^2$,

$$\lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0$$

Hence $f'(z_0) = 2z_0 \quad \forall z_0 \in \mathbb{C}$, similarly

$f''(z_0) = 2$, $f^{(n)}(z_0) = 0$, $n \geq 3$.

<u>Proposition</u> If f is differentiable at z_0, then f is continuous at z_0:

$$\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} (z - z_0)$$

$$= f'(z_0) \cdot 0 = 0 \Rightarrow \triangleright$$

The rules of differentiation are the same as for real functions

(i) $(f \pm g)' = f' \pm g'$, (ii) $(f \cdot g)' = fg' + f'g$

(iii) $(f/g)' = (f'g - fg')/(g^2)$

◁ As an example, we prove (ii)

$$(f\cdot g)' = \lim_{z\to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z-z_0}$$

$$= \lim_{z\to z_0}\left(\frac{f(z)g(z) - f(z)g(z_0)}{z-z_0} + \frac{f(z)g(z_0) - f(z_0)g(z_0)}{z-z_0}\right)$$

$$= \lim_{z\to z_0} f(z) \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}$$

$$+ g(z_0) \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f(z_0)g'(z_0) + f'(z_0)g(z_0)$$ ▷

We denote the composition of

$f: S \to \mathbb{C}$ and $g: T \to \mathbb{C}$,

where $f(S) \subseteq T$ by $g\circ f$:

$$g\circ f(z) = g(f(z))$$

Chain Rule

If f is differentiable at z_0 and g is differentiable at $f(z_0)$ then $g \circ f$ is differentiable at z_0 and

$$(g \circ f)'(z_0) = g'(f(z_0))\, f'(z_0)$$

Consider a situation when we have

$$f(z) \neq f(z_0)$$

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} \times \frac{f(z) - f(z_0)}{z - z_0}$$

Since $f(z)$ is differentiable at z_0 it is continuous there so $\lim_{z \to z_0} f(z) = f(z_0)$

giving $\lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} = g'(f(z_0))$

The Cauchy-Riemann Equations

Suppose a complex function f is written in terms of two real functions u and v of two real variables

$$f(z) = u(x,y) + i\,v(x,y),$$

$$z = x + iy$$

Next we use standard notations

$$\frac{\partial u}{\partial x}(x,y) = \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h}$$

$$\frac{\partial u}{\partial y}(x,y) = \lim_{k\to 0} \frac{u(x,y+k) - u(x,y)}{k}$$

Theorem If f is differentiable at z, then $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ all exist and

and

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

We calculate $f'(z)$ in two ways:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h \to 0} \frac{u(x+h,y) + i v(x+h,y) - u(x,y) - v(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{h \to 0} \frac{v(x+h,y) - v(x,y)}{h}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Similarly $z = x + i(y+k)$, $k \in \mathbb{R}$.

$$f'(z) = \lim_{k \to 0} \frac{u(x,y+k) + i v(x,y+k) - u(x,y) - i v(x,y)}{k}$$

$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}. \quad (i^2 = -1).$$

▷

<u>Example</u>. $f(z) = z^2 \Rightarrow f'(z) = 2z$.

$$f(z) = \underbrace{(x^2 - y^2)}_{u(x,y)} + i\underbrace{2xy}_{v(x,y)}$$

Then

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x$$

The Cauchy-Riemann Equations <u>are valid</u>.

The converse of Theorem is <u>false</u>.

If C-R conditions are satisfied at ~~it~~ this fact is not sufficient ~~to~~ to have that f is differentiable

<u>Example</u>.

$$f(x+iy)=\begin{cases} 0 & \text{if one or both of } x, y \text{ are zero} \\ 1, & \text{if } x\neq 0 \text{ and } y\neq 0 \end{cases}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0, \quad \frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0$$

<u>But</u> f is not continuous at the origin, so it cannot be differentiable there

By imposing extra continuity conditions on the partial derivatives of $u(x,y)$ and $v(x,y)$ we prove the converse theorem.

<u>Theorem</u>. If $f(z) = u(x,y) + i\,v(x,y)$ is a complex function defined on an open set S and at $z_0 = x_0 + i y_0$ the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ all exist, are continuous and satisfy C-R condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

then f is differentiable at z_0.

<u>Example</u>. $f(z) = |z|^2$ is differentiable <u>only</u> at the origin.

$$u(x,y) = x^2 + y^2, \quad v(x,y) = 0$$

Hence:

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

and the Cauchy-Riemann Equations are satisfied only at $x = y = 0$, and at this point all the partial derivatives are continuous.

Harmonic functions

Def. A function $u(x,y)$ is called harmonic if it is twice continuously differentiable and satisfies the following partial differential equation

$$\boxed{u_{xx} + u_{yy} = 0}$$

If $f(z)$ is analytic in a set S (has a derivative $\Rightarrow$ all derivatives!) then $f(z) = u(x,y) + i\,v(x,y)$ has $u(x,y)$, $v(x,y)$ that are harmonic functions on S.

$\triangle$ This is a simple consequence of the Cauchy-Riemann equations.

$$u_x = v_y \Rightarrow u_{xx} = v_{yx}$$

$$v_x = -u_y \Rightarrow u_{yy} = -v_{xy}$$

Since $v_{xy} = v_{yx} \Rightarrow$

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

A function v can be investigated similarly.

Some popular complex valued functions

$$f : \mathbb{C} \to \mathbb{C} \qquad \boxed{w = f(z)}$$

We can write the real and imaginary parts of $f(z)$

$$z = x + iy$$

$$f = u(x,y) + i\, v(x,y)$$

Example 1 (polynomial functions)

$$f(z) = z^2$$

$$z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + \underbrace{2ixy}_{v(x,y)}$$

$$f'(z) = 2z. \quad \text{(check C-R Equations)}$$

Example 2. $f(z) = e^z$

e - the base of the natural logarithm

$$e^z = e^{x+iy} = e^x e^{iy}$$

$$= e^x (\cos y + i \sin y)$$

Euler's formula

Thus:

$$u(x,y) = e^x \cos y$$

$$v(x,y) = e^x \sin y$$

Power series!

Use polar coordinates.

e

$$e^{iy} = \cos y + i \sin y$$

Take $y = \pi$

$$\boxed{e^{i\pi} + 1 = 0}$$

Formula connects 0, 1 and π

Example 3

$$f(z) = z^{1/2}$$

Find the real and imaginary parts

$$z = r e^{i(2\theta + 2k\pi)}, \; k=0,1$$

$$z^{1/2} = \sqrt{x^2+y^2}\; e^{i(\theta + k\pi)}$$

$$= \sqrt{x^2+y^2}\left(\cos(\theta+k\pi) + i\sin(\theta+k\pi)\right)$$

Thus,

$$u(x,y) = \sqrt{x^2+y^2}\,\cos(\theta + k\pi)$$

$$v(x,y) = \sqrt{x^2+y^2}\,\sin(\theta + k\pi)$$

Homework.

Find the real and imaginary parts

$$f(z) = \ln z$$

The multivalued f-tions