Lecture 4

Integration

We begin by the analysis of complex-valued functions of a real variable.

Consider a function

$$f(t) = u(t) + i\,v(t),$$

which is assumed to be a piecewise continuous function defined in the closed interval $a \le t \le b$. Such functions are integrable. The integral of $f(t)$ from $t=a$ to $t=b$, is defined as

$$\int_a^b f(t)\,dt \stackrel{\text{def}}{=} \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Simple (but important) properties of a complex integral with real variable of integration

1. $$\operatorname{Re} \int_a^b f(t)\, dt = \int_a^b \operatorname{Re} f(t)\, dt = \int_a^b u(t)\, dt.$$

2. $$\operatorname{Im} \int_a^b f(t)\, dt = \int_a^b v(t)\, dt$$

3. $$\int_a^b \left(\gamma_1 f_1(t) + \gamma_2 f_2(t)\right) dt = \gamma_1 \int_a^b f_1\, dt + \gamma_2 \int_a^b f_2\, dt$$

where $\gamma_1, \gamma_2 \in \mathbb{C}$ (any complex constants)

4.

$$\left| \int_a^b f(t)\,dt \right| \le \int_a^b |f(t)|\,dt$$

Home work. Explain the following steps: Denote

$$\theta = \operatorname{Arg}\left(\int_a^b f(t)\,dt \right).$$

Then

$$\left| \int_a^b f(t)\,dt \right| = e^{-i\theta} \int_a^b f(t)\,dt$$

$$= \int_a^b e^{-i\theta} f(t)\,dt$$

$\left| \int_a^b f(t)\,dt \right|$ is real

$$= \operatorname{Re} \int_a^b e^{-i\theta} f(t)\,dt \overset{(1)}{=} \int_a^b \operatorname{Re}\left[e^{-i\theta} f(t) \right] dt$$

$$\le \int_a^b \left| e^{-i\theta} f(t) \right| dt = \int_a^b |f(t)|\,dt$$

Example. Show that

$$\left| e^{2\alpha\pi i} - 1 \right| \le 2\pi |\alpha|$$

(No integrals are in this formulation!)

Let $f(t) = e^{i\alpha t}$, where $\alpha, t \in \mathbb{R}$.

$$\left| \int_0^{2\pi} f(t)\,dt \right| \le \int_0^{2\pi} \left| e^{i\alpha t} \right| dt$$

First step:

$$\left| \int_0^{2\pi} e^{i\alpha t}\,dt \right| = \left| \frac{e^{i\alpha t}}{\alpha} \Big|_0^{2\pi} \right| = \frac{\left| e^{2\alpha\pi i} - 1 \right|}{|\alpha|}$$

Second step

$$\int_0^{2\pi} \left| e^{i\alpha t} \right| dt = 2\pi$$

$\Longrightarrow$ $\rhd$

A contour integral

Let C be a curve, which is a set of points $z = (x, y)$ in the complex plane

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

where

$x(t)$ and $y(t)$ are continuous functions

We can write it as

$$\boxed{z(t) = x(t) + i\, y(t), \quad a \le t \le b}$$

The curve $z(t)$ is smooth, i.e. it has continuous derivative and $z'(t) \neq 0$ $\forall$ all points along the curve.

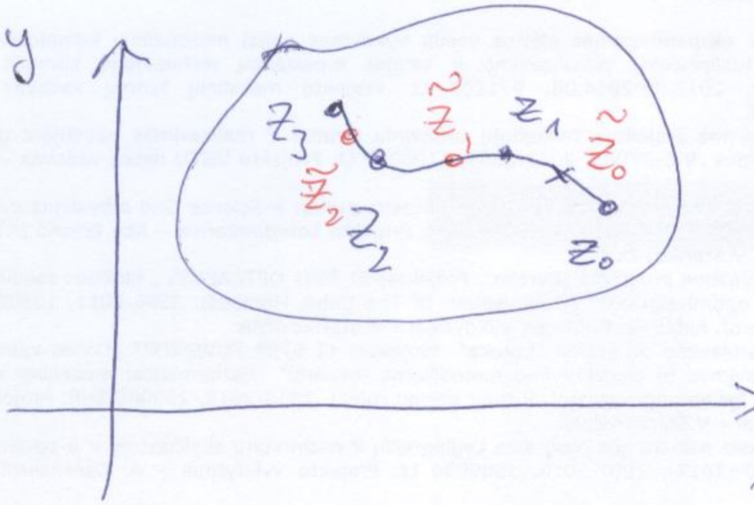
A contour is said to be a simple closed contour if the initial and final values of $z(t)$ are the same and the contour does not cross itself.

We divide the contour C into n subarcs (a standard step in the definition of integrals) by points $\boxed{z_0, z_1, \ldots, z_n = Z}$ along the direction of increasing t.

Next we form the sum

$$\sum_{k=0}^{n-1} f(\tilde{z}_k)(z_{k+1} - z_k), \quad \cancel{z - \tilde{z}_k}$$

where $\tilde{z}_k$ is an arbitrary point in the subarc $z_k z_{k+1}$.



$n = 3$

We write $\Delta z_k = z_{k+1} - z_k$

Let $\lambda = \max_k |\Delta z_k|$

Take the limit

$$\lim_{\substack{\lambda \to 0 \\ n \to \infty}} \sum_{k=0}^{n-1} f(\tilde{z}_k)\, \Delta z_k$$

If the limit exists, then the function $f(z)$ is said to be integrable along the contour C.

Let's wrote

$$\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}, \quad a \le t \le b$$

then

$$\int_C f(z)\,dz = \int_a^b f(z(t))\frac{dz}{dt}\,dt$$

$$f(z) = u(x,y) + i\,v(x,y)$$

$$dz = dx + i\,dy$$

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy$$

$$+ i\int_C u\,dy + v\,dx$$

$$= \int_a^b \left(u(x(t), y(t))\frac{dx}{dt} - v(x,y)\frac{dy}{dt}\right)dt$$

$$+ i\int_a^b \left(u(x,y)\frac{dy}{dt} + v(x,y)\frac{dx}{dt}\right)dt.$$

The usual properties of real line integrals are valid for complex integrals;

e.g.

$$\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz$$

where $(-C)$ is the opposite curve of C

<u>Example 1.</u> Evaluate the integral:

$$\oint_C \frac{1}{z-z_0}\,dz$$

where is a circle centered at z_0 and of <u>any</u> radius. The path is defined in the anticlockwise direction

The circle is parameterized by

$$z(t) = z_0 + re^{it}, \quad 0 \le t \le 2\pi.$$

The integral becomes

$$\oint_C \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{z(t)-z_0}\frac{dz}{dt}\,dt$$

$$= \int_0^{2\pi} \frac{i r e^{it}}{r e^{it}}\,dt = 2\pi i$$

(independent of r)

The result can depend on the selected path or not

(1) $\int_C |z|^2 dz$ and (2) $\int_C \frac{1}{z^2} dz$

where the contour C is

(a) the line segmen with initial point -1 and final point i.

Answ. (1) $I = \frac{2}{3}(1+i)$ (2) $-1+i$

(b) the contour C is a subarc from -1 to i, the unit circle $\operatorname{Im} z \geq 0$.

(1) Answ. $I = 1 + i$ (different value with (a)

(2)

$I = -1 + i$ (the same value!

When such a result is valid in general?

Under what conditions

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz,$$

where C_1 and C_2 are two contours in domain D with the same initial and final points and $f(z)$ is piecewise continuous inside D.

It is clear that the above question is equivalent to the question: when does

$$\oint_C f(z)\, dz = 0,$$

where C is any closed contour lying completely inside D?

$$(C = C_1 \cup (-C_2)).$$

Cauchy integral theorem

Let $f(z) = u(x,y) + i v(x,y)$ be analytic on and inside a simple closed contour C and let $f'(z)$ be also continuous on and inside C, then

$$\oint_C f(z)\, dz = 0.$$

Recall that

A function $f(z)$ is said to be analytic at some point z_0 if it is differentiable at every point of a certain neighborhood of z_0.

$f(z)$ is analytic $\Leftrightarrow \exists N(z_0; \varepsilon)$, $\varepsilon > 0$ such that $f'(z)$ exists for all $z \in N(z_0; \varepsilon)$.

Example $f(z) = |z|^2$ is differentiable only for $z_0 = 0$, hence $f(z) = |z|^2$ is not analytic at $z = 0$.

△ Let us use the Green theorem for a positively oriented closed contour C

Th. (Green's) If the two real functions $P(x,y)$ and $Q(x,y)$ have continuous first order partial derivatives on and inside C, then

$$\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dx\,dy$$

where D is the simply connected domain bounded by C.

We write:

$$f(z) = u(x,y) + i v(x,y)$$

and

$$\oint_C f(z)\,dz = \oint_C (u\,dx - v\,dy) + i \oint_C (v\,dx + u\,dy)$$

Functions $u(x,y)$, $v(x,y)$ have continuous derivatives on and inside C.

Using the Green theorem we get

$$\oint_C f(z)\,dz = \iint_D (-v_x - u_y)\,dx\,dy + i \iint_D (u_x - v_y)\,dx\,dy$$

Due to Cauchy-Riemann relations double integrals are equal to zero.

$\triangleright$

The integral of an analytic function throughout a simply connected domain D depends on the end points and not on the particular contour.

Example. C is the curve

$$y = x^3 - 3x^2 + 4x - 1$$

joining points $(1, 1)$ and $(2, 3)$.

Find the value of the integral

$$\int_C (12z^2 - 4iz)\, dz$$

The integral is independent of the path joining $(1,1)$ and $(2,3)$

$$\underset{1+i}{\parallel} \qquad\qquad 2+3i$$

$$\underline{I} = \int_{1+i}^{2+3i} (12z^2 - 4iz)\, dz$$

$$= \left(4z^3 - 2iz^2\right)\Big|_{1+i}^{2+3i} = -156 + 38i$$

<u>Case 2</u>: Compute the value of this integral along this curve:

a) a line path from $(1,1)$ to $(2,1)$

b) a line path from $(2,1)$ to $(2,3)$

a) parametrization

$y = 1$, $dy = 0$ so $z = x + i$

$dz = dx$

Cauchy integral formula

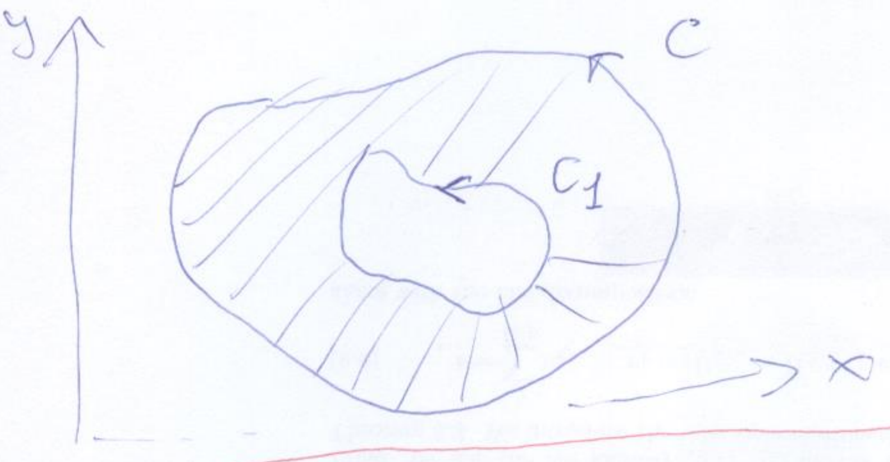
Th. Let $f(z)$ be analytic on and inside a positively oriented simple closed contour C and $\underline{z}$ is any point inside C, then

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\eta)}{\eta - z}\, d\eta .$$

◁ Take a circle C_r, with radius r around the point z, small enough to be completely inside C.

Since $\frac{f(\eta)}{\eta - z}$ analytic between C_r and C:

$$\frac{1}{2\pi i} \oint_C \frac{f(\eta)}{\eta - z}\, d\eta = \frac{1}{2\pi i} \oint_{C_r} \frac{f(\eta)}{\eta - z}\, d\eta$$

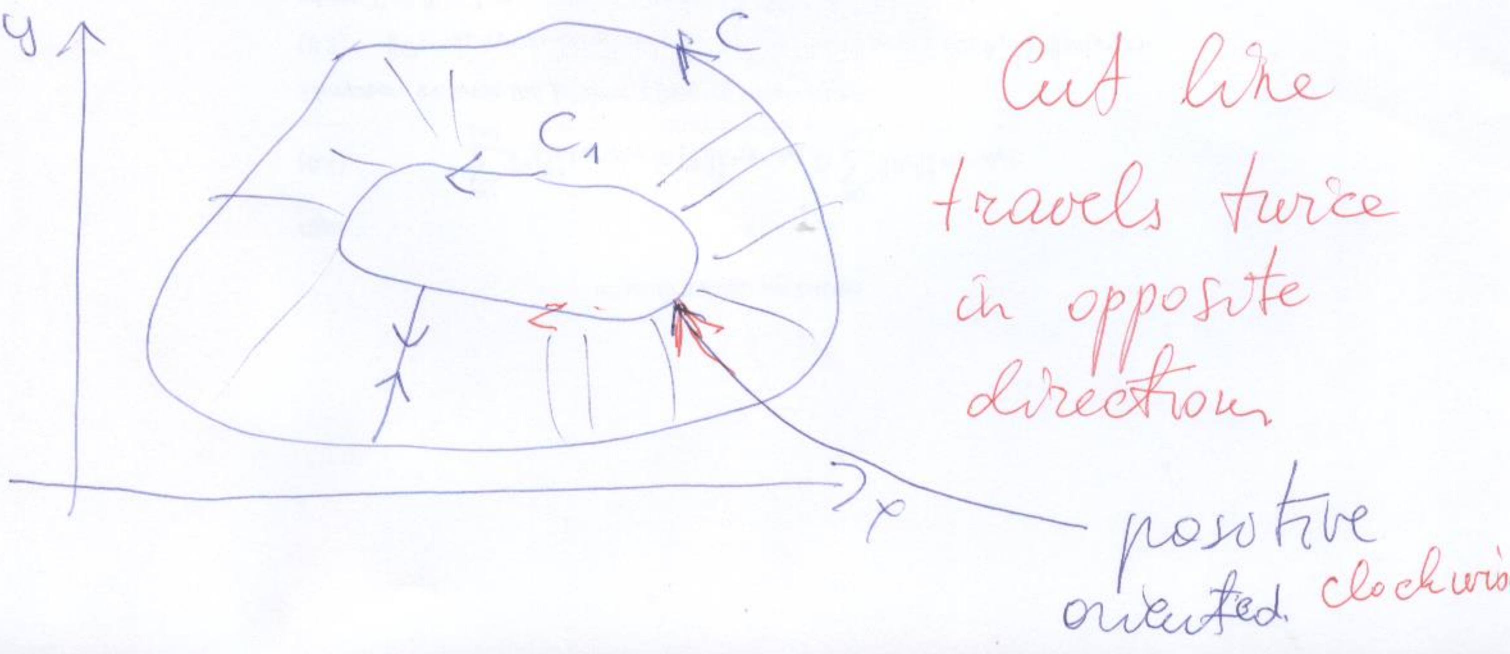


$$\boxed{\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz}$$

$f(z)$ is analytic on the set

$$S = C \cup \operatorname{int} C \setminus \operatorname{int} C_1$$

where $\underline{\operatorname{int} C_j}$ denotes the collection of all points bounded $\underline{\text{inside}}$ C_j.



For any closed contour $\tilde{C}$ and analytic function $f(z)$

$$\oint_{\tilde{C}} f(z)\,dz = 0$$

In our case

$$\tilde{C} = C \cup (-C_1) \cup W \cup -W$$

$$\oint_C f(z)\,dz + \oint_{(-C_1)} f(z)\,dz + \underbrace{\int_W f(z)\,dz + \int_{-W} f(z)\,dz}_{0} = 0.$$

Thus we get

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$$

Return back to the main proof:

$$\frac{1}{2\pi i} \oint_C \frac{f(\eta)}{\eta - z} d\eta = \frac{1}{2\pi i} \oint_{C_z} \frac{f(\eta)}{\eta - z} d\eta$$

$$= \underbrace{\frac{1}{2\pi i} \oint_{C_z} \frac{f(\eta) - f(z)}{\eta - z} d\eta}_{= 0} + \underbrace{\frac{f(z)}{2\pi i} \oint_{C_z} \frac{1}{\eta - z} d\eta}_{= f(z)}$$

(Homework!)

<u>Example</u> Evaluate:

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$$

where C is the circle

$$|z - i| = 3.$$

$$I = \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2}\,dz - \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-1}\,dz$$

Since $z=1$, $z=2$ are inside C and $\sin\pi z^2 + \cos\pi z^2$ is an analytic on and inside C, we can use Cauchy integral formula.

$$\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2} = 2\pi i\left(\sin(\pi 2^2) + \cos(\pi 2^2)\right)$$
$$= 2\pi i$$

$$\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-1}\,dz = 2\pi i\left(\sin\pi + \cos\pi\right) = -2\pi i$$

$$I = 2\pi i - (-2\pi i) = 4\pi i.$$

Derivatives of contour integrals

Suppose that it is justified a possibility ~~to~~ to differentiate both sides of the Cauchy integral formula with respect to z:

$$f'(z) = \frac{1}{2\pi i} \frac{d}{dz}\left(\oint \frac{f(\eta)}{\eta - z} d\eta \right)$$

$$= \frac{1}{2\pi i} \oint_C \left(\frac{d}{dz} \frac{f(\eta)}{\eta - z} \right) d\eta = \frac{1}{2\pi i} \oint_C \frac{f(\eta)}{(\eta - z)^2} d\eta$$

By induction we can show the generalized Cauchy formula

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_C \frac{f(\eta)}{(\eta - z)^{k+1}} d\eta, \quad k \geq 1.$$

Theorem. If a function $f(z)$ is analytic at a point, then its derivatives of all orders are also analytic at the same point.

Example. Evaluate

$$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz,$$ where C is the circle $|z| = 3$.

Let $f(z) = e^{2z}$, $a = -1$ in the CIF:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$n = 3$

$$f'''(z) = 8e^{2z} \Rightarrow f'''(-1) = 8e^{-2}$$

$$8e^{-2} = \frac{3!}{2\pi i} \oint_C \frac{e^{2z}}{(z+1)^4}\,dz.$$