Lecture 5.

Series

An infinite sequence of complex numbers $\{z_n\}$ is called a complex sequence.

<u>Convergence</u>

If for each positive quantity ε there exists a positive integer N such that

$$|z_n - z| < \varepsilon \quad \text{whenever } n > N = N(\varepsilon)$$

then the sequence is said to converge to the limit z:

$$\lim_{n \to \infty} z_n = z.$$

The limit of a convergent sequence is unique.

Suppose we write

$$z_n = x_n + i y_n, \quad z = x + i y, \text{ then}$$

$$|x_n - x| \le |z_n - z| \le |x_n - x| + |y_n - y|$$

$$|y_n - y| \le |z_n - z|$$

Suppose

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y$$

then for any $\varepsilon > 0$

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{for } n > N_x$$

$$|y_n - y| < \frac{\varepsilon}{2} \quad \text{for } n > N_y.$$

Choose $N = \max(N_x, N_y)$, then

$$|z - z_n| \le |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It is easy to show that

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y \,. \quad !$$

<u>Example</u>. The sequence

$$z_n = \frac{1}{n^3} + i, \quad n = 1, 2, \dots$$

converges to i, since

$\lim\limits_{n\to\infty} \frac{1}{n^3}$ and $\lim\limits_{n\to\infty} 1$ exist, so

$$\lim_{n\to\infty}\left(\frac{1}{n^3} + i\right) = \lim_{n\to\infty}\frac{1}{n^3} + i \lim_{n\to\infty} 1$$

$$= 0 + i \cdot 1 = i,$$

$|z_n - i| < \varepsilon$ whenever $n > \frac{1}{\sqrt[3]{\varepsilon}}$.

Def.

An infinite series of complex numbers $z_1, z_2, z_3, \ldots$ is the infinite sum, given by

$$z_1 + z_2 + \ldots = \sum_{k=1}^{\infty} z_k$$

$$\sum_{k=1}^{\infty} z_k = \lim_{n \to \infty} \sum_{k=1}^{n} z_k$$

We define the sequence of partial sums by

$$S_n = \sum_{k=1}^{n} z_k$$

If the sequence $\{S_n\}$ converges to S, then the series is said to be convergent and S is its sum, otherwise, the series is divergent

We define the remainder after n terms

$$R_n = S - S_n \left(:= \sum_{k=n+1}^{\infty} z_k \right).$$

Obviously

$$\lim_{n\to\infty} R_n = 0 \quad \text{if series is convergent}$$

Conversely, if

$$\lim_{n\to\infty} R_n = 0,$$

this is equivalent to $\forall \varepsilon > 0$

$$|S_n - S| < \varepsilon \quad \text{for } n > N(\varepsilon),$$

hence

$$S = \lim_{n\to\infty} S_n.$$

Convergence conditions.

- A necessary condition for the convergence

$$\lim_{n\to\infty} z_n = 0.$$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (R_{n-1} - R_n)$$

$$= \lim_{n\to\infty} R_{n-1} - \lim_{n\to\infty} R_n = 0.$$

- Comparison Test

Suppose

$\sum_{k=1}^{\infty} M_k$ is a convergent series

with $M_j \in \mathbb{R}$ (real numbers)

$$M_j \geq 0.$$

If for all $k > K$ we have

$$|z_k| \leq M_k \quad \forall k > K$$

Then the series

$$\sum_{k=1}^{\infty} |z_k| \quad \text{converges also.}$$

◁ $\{\tilde{S}_n\}$, where

$$\tilde{S}_n = \sum_{k=1}^{n} |z_k|$$

is a <u>bounded</u> and increasing sequence ▷

<u>Absolute convergence</u>

Def. The series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |z_n|$ converges.

Note that

$$|x_n| \leq \sqrt{x_n^2 + y_n^2}, \quad |y_n| \leq \sqrt{x_n^2 + y_2^2}$$

and both series

$$\sum_{n=1}^{\infty} |x_n| \quad \text{and} \quad \sum_{n=1}^{\infty} |y_n| \quad \text{must converge}$$

$$\Rightarrow \sum_{n=1}^{\infty} x_n \quad \text{and} \quad \sum_{n=1}^{\infty} y_n \quad \text{converge.}$$

The converse may not hold. and we have conditionally convergen series $\sum_{n=1}^{\infty} z_n$.

Examples

The series $\sum_{j=1}^{\infty} \frac{3+2i}{(j+1)^j}$ converges.

We compare ~~of~~ the series

$$\sum_{j=1}^{\infty} \frac{3+2i}{(j+1)^j} = \frac{3+2i}{8} + \frac{3+2i}{64} + \dots$$

with the convergent geometric series:

$$\sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

Since $|3+2i| = \sqrt{13} < 4$, then for $j \geq 3$

$$\left|\frac{3+2i}{(j+1)^j}\right| < \frac{4}{(j+1)^j} \leq \frac{1}{2^j}$$

Root test and Ratio test

Suppose

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L, \quad L<1$$

Then we can choose N: that for

$n \geq N$

$$\left|\frac{z_{n+1}}{z_n}\right| \leq r, \text{ where } L<r<1$$

Now we have

$$|z_{N+1}| \leq r\,|z_N|$$

$$|z_{N+2}| \leq r\,|z_{N+1}| \leq r^2\,|z_N|, \ \ldots$$

$$|z_{N+1}| + |z_{N+2}| + |z_{N+3}| + \ldots \leq \underbrace{|z_N|\left(r + r^2 + r^3 + \ldots\right)}_{\text{convergent series}} \text{ "} M_n$$

Thus $\sum_{n=1}^{\infty} |z_n|$ converges absolutely

by the comparison test

Sequences of complex functions

Let $f_1(z), f_2(z), \ldots, f_n(z), \ldots$ denoted by $\{f_n(z)\}$ be a sequence of complex functions defined in a region R.

Take some point $z_0 \in R$, then we get a sequence of complex numbers $\{f_n(z_0)\}$. Suppose it converges to a unique limit.

$$\boxed{f(z_0) = \lim_{n\to\infty} f_n(z_0)}$$

If this holds for $\forall z \in R$ we get a complex function $f(z)$ $z \in R$:

$$f(z) = \lim_{n\to\infty} f_n(z).$$

Definition of this converges

$\{f_n(z)\}$ converge to function $f(z)$,

$z \in R \iff$ (if and only if)

$\forall \varepsilon > 0$ we can find a positive

integer $N(\varepsilon; z)$ such that

$|f(z) - f_n(z)| < \varepsilon$ for all $n > N(\varepsilon; z)$

Def If it is possible to find

a single $N(\varepsilon)$ $\forall z \in R$, $\{f_n(z)\}$ is

said to converge uniformly to $f(z)$

in R.

Infinite series of complex function

$$f_1(z) + f_2(z) + \dots = \sum_{k=1}^{\infty} f_k(z)$$

The sequence of partial sums $\{S_n(z)\}$

$$S_n(z) = \sum_{k=1}^{n} f_k(z)$$

Def. The infinite series is said to be convergent if

$$\lim_{n\to\infty} S_n(z) = S(z),$$

where $S(z)$ is called the sum.

The properties of convergence are related to convergence of their counterparts of complex numbers.

Example Consider the complex series

$$\sum_{k=1}^{\infty} \frac{\sin kz}{k^2}$$

(a) When z is real, we have

$$\left|\frac{\sin kz}{k^2}\right| \le \frac{1}{k^2} \quad \forall k \in \mathbb{N}.$$

Since $\sum_{k=1}^{\infty} 1/k^2$ is convergent, then $\sum_{k=1}^{\infty} \sin(kz/k^2)$ is absolutely convergent for all z by the <u>comparison test</u>

(ii) Let $z = x + iy$ (non-real) $y \neq 0$

$$\left| \frac{\sin kz}{k^2} \right| \cancel{\geq} \frac{e^{-ky} e^{ikx} - e^{ky} e^{-ikx}}{2k^2 i}$$

$$e^{ikz} = e^{ik(x+iy)} = e^{ikx} e^{-ky}$$

$$e^{-ikz} = e^{-ikx} e^{ky}.$$

$$2i \cdot \sin kz = e^{ikz} - e^{-ikz}$$

$$\sin kz = \frac{1}{2i} \left(e^{-ky} e^{ikx} - e^{ky} e^{-ikx}\right)$$

We get that

$$\left|\frac{\sin kz}{k^2}\right| = \frac{1}{2k^2}\left|e^{-ky}e^{ikx} - e^{ky}e^{-ikx}\right|$$

$$\geq \frac{1}{2k^2}\left(e^{k|y|} - e^{-k|y|}\right) \xrightarrow[k\to\infty]{} \infty$$

The condition

$$\lim_{k\to\infty}\left|\frac{\sin kz}{k^2}\right| \to 0 \quad \text{is not satisfied}$$

and the series is divergent

Uniform convergence of $\sum_{k=1}^{\infty} f_k(z)$

Def. This series converges uniformly to $S(z)$ in some region R iff ($\Leftrightarrow$)

$\forall \varepsilon > 0$, $\exists N(\varepsilon)$, but not dependent on z such that $\forall z \in R$

$$|R_n(z) = S(z) - \sum_{k=1}^{n} f_k(z)| < \varepsilon, \quad n > N.$$

Weierstrass M-test

If $|f_k(z)| \leq M_k$, where M_k is independent of $z \in R$ and the series

$$\sum_{k=1}^{\infty} M_k \text{ converges,}$$

then $\sum_{k=1}^{\infty} f_k(z)$ is uniformly convergent in R.

Proof follows from the definition and properties of series of real numbers.

Examples.

$$\sum_{n=1}^{\infty} \frac{z}{n\sqrt{n+1}}, \quad |z| \leq 1$$

Note that

$$|f_n(z)| = \frac{|z|^n}{n\sqrt{n+1}} \leq n^{-3/2} \text{ if } |z| \leq 1.$$

Take $M_n = n^{-3/2}$ and $\sum M_n$ converges

Home task:

Show that the geometric series

$$\sum_{n=0}^{\infty} z^n$$

converges uniformly to $\frac{1}{1-z}$ on any closed subdisk $|z| \leq r < 1$ of the open unit disk $|z| < 1$.

Power series

Def. $$\sum_{n=1}^{\infty} a_n (z-z_0)^n$$

Then there exists a real number $R \geq 0$, such that the power series converges absolutely for $|z-z_0| < R$ and diverges for $|z-z_0| > R$. R is called the radius of convergence.

$|z-z_0| = R$ is called the circle of convergence.

Ratio test

The radius of convergence R is by

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \text{, if the limit exists.}$$

Ratio test (general def)

$$\boxed{\lim_{n \to \infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| < 1.}$$

$$f_n(z) = a_n (z-z_0)^n$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \frac{|z-z_0|^{n+1}}{|z-z_0|^n} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |z-z_0|$$

Then it is clear, that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |z-z_0| < 1$$

$$\Longleftrightarrow \quad |z-z_0| < \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

<u>Examples</u>. Find the circle of convergence

$$\sum_{k=1}^{\infty} \frac{1}{k} (z-i)^k$$

$z_0 = i$.

$$R = \lim_{k\to\infty} \frac{1/k}{1/(k+1)} = 1$$

so the circle of convergence is

$$\boxed{|z-i| = 1.}$$