Lecture 6

[Series of Functions
Taylor and Laurent series]

Find the circle of convergence

$$\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{k^2} z^k$$

First note. The ratio test is not efficient approach for this series

Lets try $\boxed{\text{The Root Test}}$ to the infinite power series.

a) The root test

Limit superior

Consider a sequence $\{x_n\}$ of real numbers $x_n \in \mathbb{R}$, and let S denote the set of all its limit points.

Example

$$x_n = 3 + (-1)^n, \quad n = 1, 2, \cdots$$

The limit points are 2 and 4.

Def. The limit superior of $\{x_n\}$ is the supremum (the least upper bound) of S.

Ex. $\overline{\lim_{n\to\infty}}\, x_n = \max(2,4) = 4.$

Root test

Suppose the limit superior of $\{|z_n|^{1/n}\}$ equals L, the the series

$$\sum_{n=1}^{\infty} z_n$$

convergence absolutely if $L<1$ and diverges if $L>1$. No info when $L=1$.

$$\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{k^2} z^k.$$

By the root test, we have

$$\frac{1}{R} = \overline{\lim_{k\to\infty}} \sqrt[k]{\left(1+\frac{1}{k}\right)^{k^2}}$$

$$= \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^{k} = e \, (!)$$

The circle of convergence is

$$|z| = \frac{1}{e}.$$

Taylor series

Suppose we have a power series, and it represents the function $f(z)$ inside the circle of convergence

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

The theory of power series gave us the following result: p. series can be differentiated termwise

$$f'(z) = \sum_{n=1}^{\infty} n\, a_n (z-z_0)^{n-1}$$

$$f''(z) = \sum_{n=2}^{\infty} n(n-1)\, a_n (z-z_0)^{n-2}, \ldots$$

Putting $z = z_0$, we get:

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, \ldots$$

Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n.$$

This function is an analytic function inside its circle of convergence.

Opposite question:

We have an analytic function $f(z)$ in $\mathcal{D}$.

Can we expand it in Taylor series and how the domain of analyticity is related ~~to~~ the circle of convergence

Taylor series theorem

Let $f(z)$ be analytic in a domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$ and $z_0 \in \mathcal{D}$. Determine R such that

$$R = \min \{ |z-z_0| , \ z \in \partial\mathcal{D} \}$$

Then there exists a power series

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k$$

which converges ~~to~~ $f(z)$ for $|z-z_0| < R$.

The coefficients are given by

$$a_k = \frac{f^{(k)}(z_0)}{k!} \left(= \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{k+1}} \right),$$

C is any closed contour around z_0 and lying completely inside $\mathcal{D}$.

Example.

Consider the function $\frac{1}{1-z}$, the Taylor series at $z=0$ is given by

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z|<1.$$

The radius of convergence can be found by the ratio test:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = 1$$

2.

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

(no singular points!)

What is the interval of convergence?

The Taylor series converges only for $|x| < 1$.

This is because the complex extension $\frac{1}{1+z^2}$ has singularities on the circle $|z| = 1$ (e.g. $z = i$, $z^2 = -1$)

The radius of convergence of

$$\sum_{n=0}^{\infty} (-1)^n z^{2n}$$

is <u>one</u>.

Laurent series

Consider an infinite power series with negative power terms

$$\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}.$$

How to find the region of convergence?

1. Set $W = \dfrac{1}{z-z_0}$

The series is defined as (a Taylor series)

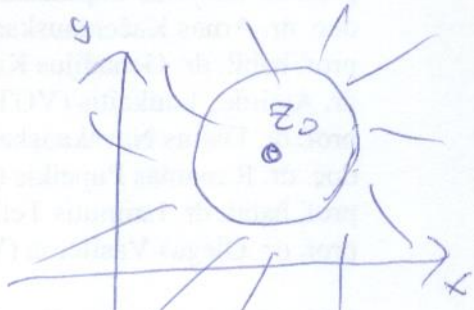
$$\sum_{n=1}^{\infty} b_n W^n$$

Suppose:

$$R' = \lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right|$$

$$\sum_{n=1}^{\infty} b_n w^n \text{ converges for } |w| < R'$$

or $$|z - z_0| > \frac{1}{R'}$$



If $R' = 0$, then the infinite series

$$\sum_{n=1}^{\infty} b_n (z - z_0)^{-n} \text{ does not converge}$$

for any z (not even at $z = z_0$)

If $$\frac{1}{R'} = \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = 0$$

we get that

$$\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| |w| < 1 \quad \forall w$$

By the ratio test the region of convergence is the whole complex plane except at $z=z_0$, that is

$$|z-z_0|>0.$$

Lauren series at z_0

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}}_{\text{the principal part}}$$

Suppose

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|, \quad R'=\lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|$$

and $RR'>1$

Then inside the domain

$$\left\{ z : \frac{1}{R'} < |z - z_0| < R \right\}$$

The Laurent series is convergent

Why $\underline{RR' > 1}$?

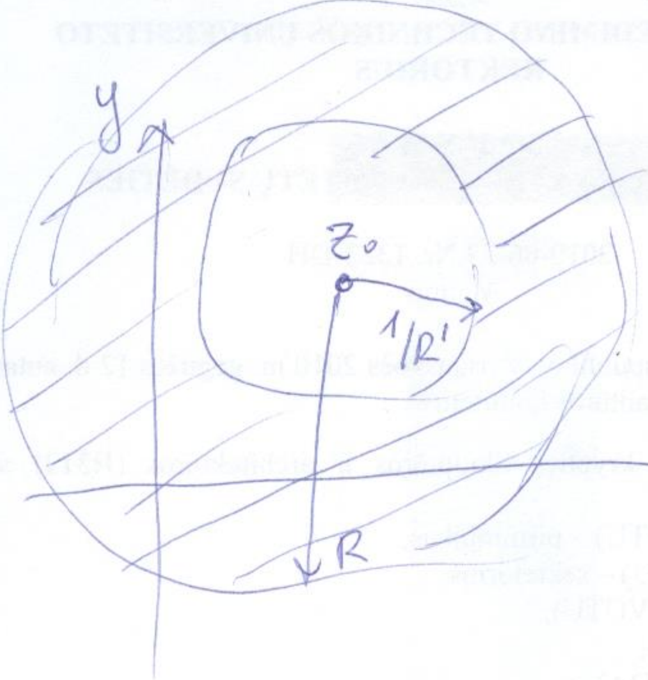
If $RR' \le 1$, then the intersection of the two regions:

$$|z - z_0| < R \text{ and } |z - z_0| > \frac{1}{R'}$$

is the empty set:

$$|z - z_0| > \frac{1}{R'} \text{ and } RR' \le 1$$

implies $|z - z_0| \ge R$, which <u>contradicts</u> with $|z - z_0| < R$.



Annular domain (a double connected domain)

<u>Theorem</u>. Let $f(z)$ be analytic in the annulus A:

$$R_1 < |z - z_0| < R_2,$$

then $f(z)$ can be represented by the Laurent series, (convergent in A)

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k$$

the coefficients are given by (C is a closed contour in A)

$$c_k = \frac{1}{2\pi i} \oint_C \frac{f(\tilde{z})}{(\tilde{z} - z_0)^{k+1}} \, d\tilde{z}, \quad k = 0, \pm 1, \ldots$$

1. If $f(z)$ is analytic in the full disc: $|z-z_0| < R_2$, then the integrand in calculating c_k, $k=-1,-2,\cdots$ becomes analytic, hence $\boxed{c_k = 0,\ k=-1,-2,\dots}$

The Laurent series is reduced to a Taylor series.

Example. Consider at $z=0$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$$

Function $e^{1/z}$ is analytic everywhere except at $z=0$

So the annulus of convergence is $|z| > 0$.

More examples

Example (continuation)

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n}$$

We observe

$$\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \to \infty} \frac{1/(n+1)!}{1/n!} = 0$$

so $\frac{1}{R} = 0$

Find c_{-1}

$$c_{-1} = \frac{1}{2\pi i} \oint_C f(\tilde{z})\, d\tilde{z}$$

Let's take the contour C as

$|z| = 1.$

and C lies completely inside the punctured disc $|z| > 0$.

Classification of singularities.

<u>Def.</u> Singular point of $f(z)$ is a point at which $f(z)$ is not analytic.

<u>Isolated singularity</u>

Existence of a neighborhood of z_0 in which z_0 is <u>the</u> only singular point of $f(z)$.

Example.

$f(z) = \frac{1}{z^2+1}$ has $\pm i$ as isolated singularities

Non-isolated singularities

$f(z) = \bar{z}$ is nowhere analytic so that every point in $\mathbb{C}$ is a non-isolated singularity.

Suppose z_0 is an isolated singularity $\Rightarrow$ there exists $R > 0$ such that $f(z)$ is analytic inside the deleted neighborhood

$$0 < |z - z_0| < R.$$

In this annular domain the Laurent series theorem is applicable

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

$$0 < |z - z_0| < R.$$

A classification of isolated singular points z_0 (use the principal part of L series

It is either:

- removable singularity
- essential singularity
- pole

Removable singularity.

The principal part vanishes and the Laurent series is essentially a Taylor series.. The series represent an analytic function in $|z - z_0| < R$.

Example.

$$\frac{1-\cos z}{z^2}$$ is not defined at $z=0$

The Laurent expansion (in a deleted neighborhood of $z=0$)

$$\frac{1-\cos z}{z^2} = \frac{1}{z^2}\left(1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)\right)$$

$$= \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots$$

$$0<|z|<\infty$$

Since

$$\lim_{z\to 0}\frac{1-\cos z}{z^2} = \frac{1}{2!}$$ we can

remove the singularity at $z=0$; by defining

$$f(0)=\frac{1}{2} \quad \Big| \quad f(z)=\begin{cases}(1-\cos z)/z^2 & z\neq 0\\ 1/2, & z=0\end{cases}$$

Taylor series is valid for $|z|<\infty$

Essential singularity

The principal part has <u>infinitely</u> many non-zero terms

<u>Example</u> The Laurent series

$$z^2 e^{1/z} = z^2 + z + \frac{1}{2!} + \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \cdots, \quad 0 < |z| < \infty$$

<u>Pole of order k</u>

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_k}{(z-z_0)^k}$$

$b_k \neq 0$

It is called a simple pole when $k=1$.

$$\left(f(z) = \frac{1}{1+z^2}, \quad z = \pm i \right).$$

Practical work

Find the order of a pole

1. $\dfrac{1-\cos z}{z^5}$ (answ. 3 at $z=0$)

2. $f(z) = \dfrac{1}{z(z-1)^2}$ (answ. 2 at $z=1$)

why $z=1$ is not an essential singularity.

Find the annulus of convergence.