Practical work Lect 6

Example 1

Find the Laurent expansion of

$$f(z) = \sin\left(z - \frac{1}{z}\right)$$

The function has singularity at

$z = 0$.

So the annulus of convergence is

$$|z| > 0$$

The coefficient c_n is given by

$$c_n = \frac{1}{2\pi i} \oint_C \frac{\sin\left(z - \frac{1}{z}\right)}{z^{n+1}} \, dz, \quad n = 0, \pm 1, \pm 2, \ldots$$

C is chosen to be the unit circle

$$|z| = 1$$

We take $z = e^{i\theta}$ so

$$dz = ie^{i\theta}\, d\theta$$

Then

$$c_n = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{\sin(2i \sin\theta) ie^{i\theta}}{e^{i(n+1)\theta}} d\theta$$

$$\boxed{= \frac{1}{2\pi} \int_{-\pi}^{\pi} i \sinh(2\sin\theta)\Big(\cos(n\theta) - i\sin(n\theta)\Big) d\theta}$$

$$\sinh x = -i \sin(ix).$$

$$\parallel \qquad\qquad e^{ix} = \cos x + i \sin x.$$

$$\frac{e^x - e^{-x}}{2} \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$

$$x \in \mathbb{R}. \qquad \sinh(ix) = \frac{e^{ix} - e^{-ix}}{2}$$

$$= i \sin x$$

Taylor

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$\sinh(2\sin\theta)$ is an odd function

in θ $\Rightarrow$

$$\int_{-\pi}^{\pi} \sinh(2\sin\theta)\cos n\theta \, d\theta = 0.$$

Thus

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sinh(2\sin\theta)\sin(n\theta)\, d\theta$$

$n = 0, \pm 1, \pm 2, \ldots$

Example 2

Find all the possible Taylor and Laurent series expansions of

$$f(z) = \frac{1}{(z-i)\,(z-2)} \quad \text{at } z_0 = 0.$$

Specify the regions of convergence

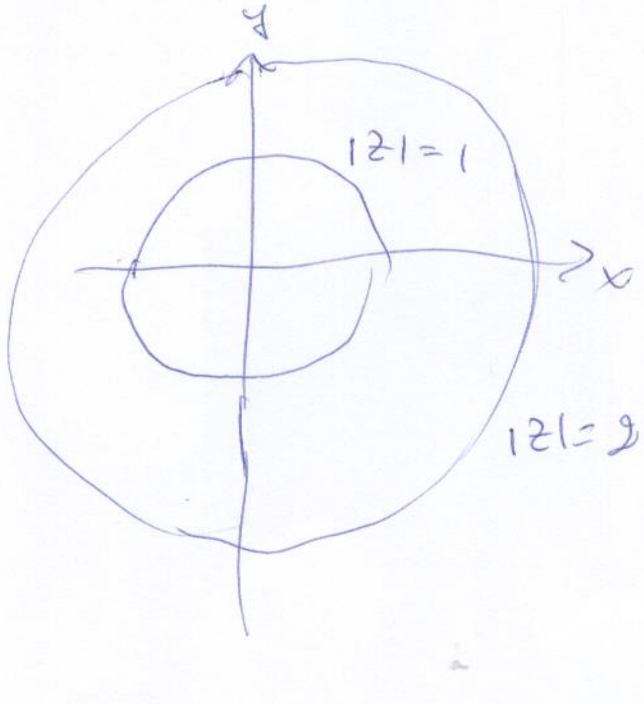
Two isolated singularities at

$z = i$ and $z = 2$

The possible stability domains

1. $|z| < 1$ 2. $1 < |z| < 2$

3. $|z| > 2$



1) For $|z| < 1$.

$$\frac{1}{z-i} = \frac{i}{1-\frac{z}{i}} =$$

$$= i\left[1 + \frac{z}{i} + \left(\frac{z}{i}\right)^2 + \cdots + \frac{z^n}{i^n} + \cdots\right]$$

2) For $|z| < 2$

$$\frac{1}{z-2} = \left(-\frac{1}{2}\right)\frac{1}{1-\frac{z}{2}}$$

$$= \left(-\frac{1}{2}\right)\left[1 + \frac{z}{2} + \cdots + \left(\frac{z}{2}\right)^n + \cdots\right]$$

$$f(z) = \frac{1}{i-2}\left(\frac{1}{z-i} - \frac{1}{z-2}\right)$$

$$= \frac{1}{i-2}\left[i \sum_{n=0}^{\infty} \left(\frac{z}{i}\right)^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \right]$$

$$= \frac{1}{i-2} \sum_{n=0}^{\infty} \left[\left(\frac{1}{i}\right)^{n-1} + \frac{1}{2^{n+1}} \right] z^n .$$

This is a Taylor series and converges inside the solid disc $|z| < 1$.

(2) For $1 < |z| < 2$

$$\frac{1}{2-i} = \frac{1}{z} \frac{1}{1-\frac{i}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{i}{z}\right)^n \qquad |z| > 1$$

$N \to N$

$$\frac{1}{z-2} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \quad |z|<2$$

$$f(z) = \frac{1}{i-2}\left[\sum_{n=0}^{\infty}\left(\frac{i^n}{z^{n+1}} + \frac{z^n}{2^{n+1}}\right)\right]$$

$1<|z|<2.$

(3) $|z|>2$

$$\frac{1}{z-i} = \sum_{n=0}^{\infty} \frac{i^n}{z^{n+1}}, \quad |z|>1$$

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \quad |z|>2$$

$$\Rightarrow f(z) = \frac{1}{i-2}\left[\sum_{n=0}^{\infty}(i^n - 2^n)\frac{1}{z^{n+1}}\right]$$

$|z|>2$