Lecture 7

Simple method of finding the order of a pole.

If z_0 is a pole of order $\underline{k}$, then

$$\lim_{z \to z_0} (z - z_0)^k f(z) = b_k, \quad b_k \neq 0$$

In general

$$\lim_{z \to z_0} (z - z_0)^m f(z) = \begin{cases} b_k, & m = k \\ 0, & m > k \\ \infty, & m = k \end{cases}$$

Ex. $f(z) = (1 - \cos z) / z^5$

$$\lim_{z \to 0} z^m \frac{1 - \cos z}{z^5} = \begin{cases} 1/2, & m = 3 \\ 0, & m > 3 \\ \infty, & m < 3 \end{cases}$$

Hence $z = 0$ is a pole of order 3 of $f(z)$

Residue calculus

Let z_0 be an isolated singularity of $f(z)$.

Then there exists a deleted neighborhood

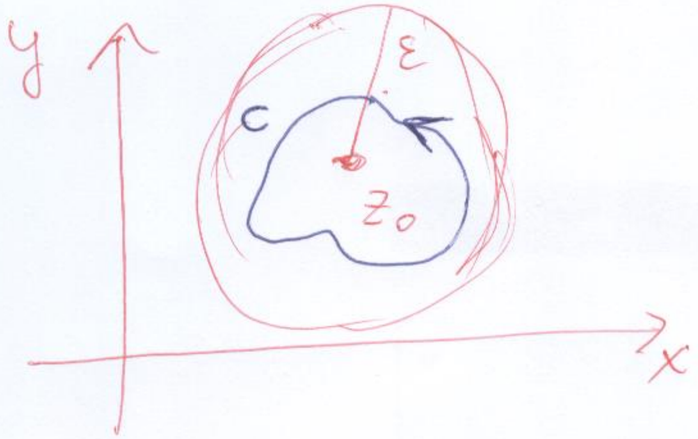
$$N_\varepsilon = \{ z : 0 < |z - z_0| < \varepsilon \}$$

such that f is analytic inside N_ε.

We define

$$\boxed{\mathrm{Res}(f, z_0) = \frac{1}{2\pi i} \oint_C f(z)\,dz}$$

where C is any simple closed contour around z_0 and inside N_ε.



Since $f(z)$ has a Laurent expansion inside N_ε

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

then
$$\boxed{b_n = \frac{1}{2\pi i} \oint_C \frac{f(\tilde{z})}{(\tilde{z}-z_0)^{-n+1}}\, d\tilde{z}}$$

$$b_1 = \operatorname{Res}(f, z_0).$$

Ex. $$\operatorname{Res}\left(\frac{1}{(z-z_0)^k}, z_0\right) = \begin{cases} 1, & \text{if } k=1 \\ 0, & \text{if } k \neq 1 \end{cases}$$

$$e^{1/z} = 1 + \frac{1 \cdot 1}{1!\, z} + \frac{2}{2!} \frac{1}{z^2} + \dots$$

$|z| > 0$

$$\Rightarrow \operatorname{Res}\left(e^{1/z}, 0\right) = 1.$$

Ex. Find

$$\operatorname{Res}\left(\frac{1}{(z-1)(z-2)}, 1\right)$$

by using the Cauchy integral formula

$$\boxed{f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz}$$

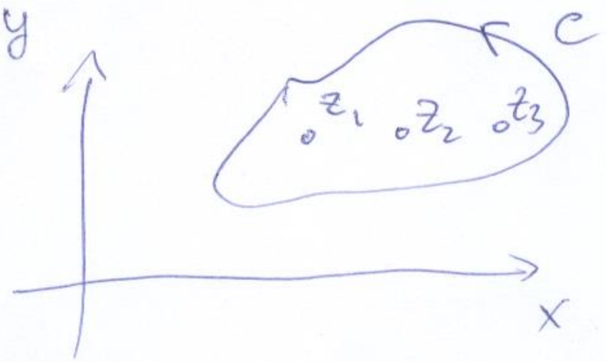
C is a circle

oriented counterclock-wise

Cauchy residue theorem

$f(z)$ is analytic inside domain defined by a closed contour C except at the isolated singularities

$z_1, z_2, \ldots, z_n$.



$$\oint_C f(z)\, dz = 2\pi i \left[\sum_{k=1}^{n} \operatorname{Res}(f, z_k) \right]$$

Example

$$\oint_{|z|=1} \frac{z+1}{z^2}\, dz.$$

Use:

(i) direct contour integration

$$z = e^{i\theta} \qquad 0 < \theta \leq 2\pi$$

$$dz = i e^{i\theta} d\theta$$

$$\oint_{|z|=1} \frac{z+1}{z^2} dz = \int_0^{2\pi} \left(e^{-i\theta} + e^{-2i\theta}\right) i e^{i\theta} d\theta$$

$$= i \int_0^{2\pi} \left(1 + e^{-i\theta}\right) d\theta = \underline{2\pi i.}$$

(ii) $f(z) = \dfrac{z+1}{z^2}$ has a double pole at $z = 0$.

The Laurent expansion in a deleted neighborhood of $z = 0$ is

$$f(z) = \frac{1}{z} + \frac{1}{z^2}. \Rightarrow \boxed{b_1 = 1.}$$

$$\operatorname{Res}\left(\frac{z+1}{z^2}, 0\right) = 1$$

$\Rightarrow$

$$\oint_{|z|=1} \frac{z+1}{z^2}\, dz = 2\pi \operatorname{Res}\left(\frac{z+1}{z^2}, 0\right) \boxed{= 2\pi i}$$

Theory (algorithm)

Let z_0 be a pole of order k.

In a deleted neighborhood of z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \dots + \frac{b_k}{(z - z_0)^k}, \quad b_k \neq 0.$$

Consider $g(z) = (z-z_0)^k f(z)$

$$g(z) = b_k + b_{k-1}(z-z_0) + \dots + b_1 (z-z_0)^{k-1}$$

$$+ \sum_{n=0}^{\infty} a_n (z-z_0)^{n+k}$$

(the principal part of $g(z)$ vanishes):

$$b_1 = \mathrm{Res}(f, z_0) = \begin{cases} \dfrac{g^{(k-1)}(z_0)}{(k-1)!} \end{cases}$$

if $g^{(k-1)}(z)$ is analytic at z_0

$$= \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{(z-z_0)^k f(z)}{(k-1)!} \right.$$

if z_0 is a removable singularity of $g^{(k-1)}(z)$

Example 1.

Simple pole $k=1$.

$$\operatorname{Res}(f, z_0) = \lim_{z \to 0} (z-z_0) f(z).$$

Suppose

$$f(z) = \frac{p(z)}{q(z)}, \quad p(z_0) \neq 0, \quad q(z_0) = 0$$

$$q'(z_0) \neq 0$$

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z-z_0) \frac{p(z_0) + p'(z_0)(z-z_0) + \cdots}{q'(z_0)(z-z_0) + \frac{q''}{2!}(z-z_0)^2 + \cdots}$$

$$= \frac{p(z_0)}{q'(z_0)}.$$

Find the residue of (practical work)

$$f(z) = \frac{e^{1/z}}{1-z}$$

at all isolated singularities

(i) simple pole $z = 1$.

(i) essential singularity a $z = 0$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots$$

Solution

$$\operatorname{Res}(f, 1) = \lim_{z \to 1} (z-1)\, f(z)$$

$$= - e^{1/z}\Big|_{z=1} = -e.$$

Consider

$$\frac{e^{1/z}}{1-z} = \left(1 + z + z^2 + \cdots\right)\left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots\right)$$

for $0 < |z| < 1$

The coefficient b_1: $\left(= \operatorname{Res}(f, 0) \right)$

$$b_1 = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e - 1$$

$$\left(e = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \right).$$

$$\operatorname{Res}(f, 0) = e - 1.$$

Example

Evaluate

$$\oint_{|z|=2} \frac{5z-2}{z(z-1)}\,dz$$

$$f(z) = \frac{5z-2}{z(z-1)}$$

$$= \frac{5z-2}{z} \cdot \frac{(-1)}{1-z} = \left(5-\frac{2}{z}\right)(-1)\left(1+z+z^2+\ldots\right.$$

$$\Rightarrow \operatorname{Res}(f, 0) = 2$$

For $0 < |z-1| < 1$

$$\frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z(z-1)} = \frac{5(z-1)+3}{z-1} \times$$

$$\times \frac{1}{1+(z-1)}$$

$$= \left(5 + \frac{3}{z-1}\right) \times \left(1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots\right)$$

so that

$$\mathrm{Res}(f, 1) = 3.$$

$$\oint_{|z|=2} \frac{5z-2}{z(z-1)}\, dz = 2\pi i \left[\mathrm{Res}(f,0) + \mathrm{Res}(f,1)\right]$$

$$= 10\pi i.$$

Calculation of real valued integrals

1. Integrals of trigonometric functions

We consider a real integral

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

$R(x, y)$ - is a rational function defined inside the unit circle

$|z| = 1$, $z = x + iy$

Our aim is to convert this integral into a contour integral around the unit circle.

We apply the following substitutions

$$z = e^{i\theta}, \quad dz = i e^{i\theta} d\theta = i z d\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \boxed{\frac{1}{2}\left(z + \frac{1}{z}\right)}$$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

$$= \oint_{|z|=1} \frac{1}{iz} R\left(\frac{z + 1/z}{2}, \frac{z - z^{-1}}{2i}\right) dz$$

A magic step

$$= 2\pi i \sum \operatorname{Res}\left(\frac{1}{iz} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right), \text{ inside } |z| = 1\right)$$

Example

Compute

$$I = \int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta}\, d\theta$$

$$I = -i \oint_{|z|=1} \frac{\frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)}{2 + \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{z}$$

$$= -i \oint_{|z|=1} \underbrace{\frac{z^4+1}{z^2(z^2+4z+1)}}_{f(z)} \, dz.$$

$f(z)$ has a pole of order two at $z=0$.

The roots of z^2+4z+1

i.e. $z_1 = -2+\sqrt{3}$, $z_2 = -2-\sqrt{3}$

are simple poles of $f(z)$

Note that

z_1 is inside but z_2

is outside $|z|=1$.

$$\mathrm{Res}\,(f,0) = \lim_{z\to 0} \frac{d}{dz} \frac{z^4+1}{z^2+4z+1} = -4$$

$$\mathrm{Res}\,(f, -2+\sqrt{3}) = \quad \boxed{\begin{array}{l} g(z) = (z-z_0)^k f(z) \\ b_1 = c_{-1} = \dfrac{g^{(k-1)}(z_0)}{(k-1)!} \end{array}}$$

$$= \left.\frac{z^4+1}{z^2}\right|_{z=-2+\sqrt{3}} \Big/ \left.\frac{d}{dz}\left(z^2+4z+1\right)\right|_{z=-2+\sqrt{3}}$$

$$\boxed{\mathrm{Res}\left(\frac{p(z)}{q(z)}, z_0\right) = \frac{p(z_0)}{q'(z_0)}} \qquad = \frac{7}{3}$$

$$I = (-i)\, 2\pi i \left[\mathrm{Res}\,(f,0) + \mathrm{Res}\,(f, -2+\sqrt{3})\right)$$

$$= 2\pi \left(-4 + \frac{7}{\sqrt{3}}\right).$$

Cauchy residue theorem

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}\,(f, z_j)$$

$z_1, z_2, \dots, z_n$ are the isolated singularities