Lecture 8

Fourier series

Let us start with $\boxed{\sin x}$ f-unction

1) It has period 2π

$$\sin(x+2\pi) = \sin x.$$

2) It is odd function

$$\sin(-x) = -\sin x$$

3) Boundary conditions

$$\sin(0) = 0, \quad \sin\pi = 0.$$

$\cos(\pi n x)$

$\sin(\pi n x)$

. . . . .

A Fourier series is an example of trigonometric series, e.g.

$$\sum_{n=1}^{\infty} c_n \sin(\pi n x),$$

but not all trigonometric are Fourier series!

Fourier series, sine-cosine form

$$S_N(x) = A_0 + \sum_{n=1}^{N}{}' \left(A_n \cos\left(2\pi \frac{n}{P} x\right) + B_n \sin\left(2\pi \frac{n}{P} x\right)\right)$$

where P defines the period, $N \to \infty$ (in a limit).

Fourier series, exponential form

$$S_N(x) = \sum_{n=-N}^{N}{}' C_n e^{i 2\pi \frac{n}{P} x}$$

We call harmonics, they are indexed by an integer n (the number of cycles in interval [0, P]).

A wave length is equal to $\frac{P}{n}$

A frequency is equal to $\frac{n}{P}$.

Fourier series analysis

We take a given <u>real-valued</u> function $f(x)$ (x - can represent a time).

$$A_0 = \frac{1}{P} \int_P f(x)\,dx$$

$$A_n = \frac{2}{P} \int_P f(x) \cos\left(2\pi \frac{n}{P} x\right) dx, \quad n \geq 1$$

$$B_n = \frac{2}{P} \int_P f(x) \sin\left(2\pi \frac{n}{P} x\right) dx, \quad n \geq 1$$

P denotes the chosen interval : e.g

$[0, P]$, $[-\frac{P}{2}, \frac{P}{2}]$.

In place of separate formulas for a_0 and a_n and b_n we can get $\boxed{\text{one formula}}$ for all the <u>complex coefficients</u> c_n.

<u>Complex Fourier series</u>

$$F(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \left(\text{or } e^{i\pi n x} \right)$$

$$= c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + \dots$$

<u>1 conclusion</u> If every $c_n = c_{-n}$ we can combine

$$e^{inx} + e^{-inx}$$

$$= (\cos nx + i \sin nx) + (\cos nx - i \sin nx)$$

$$= 2 \cos nx. \text{ (cosine series)}$$

Conclusion 2.

Show that if every

$$c_n = -c_{-n}$$

then

$$e^{inx} - e^{-inx} = 2i \sin nx$$

then we get the sine series for an odd function

To find c_n multiply series(*) by e^{-inx} and integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} F(x) e^{-inx}\, dx = \sum_{k=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{+ikx} \cancel{dx}\, e^{-inx}\, dx$$

$$= 2\pi c_n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Homework Check this formula for C_0, take into accocunt that $e^0 = 1$.

Complex inner product

$$(F, G) = \int_{-\pi}^{\pi} F(x)\,\overline{G}(x)\,dx$$

Orthogonality of e^{inx} and e^{ikx}:

$n \neq k$.

$$\int_{-\pi}^{\pi} e^{inx}\, e^{-ikx}\, dx = \int_{-\pi}^{\pi} e^{i(n-k)x}\, dx$$

$$= \frac{1}{i(n-k)}\, e^{i(n-k)x} \Big|_{-\pi}^{\pi} = 0.$$

A. Example 1.

Add the complex series for

$$\frac{1}{2-e^{ix}} \quad \text{and} \quad \frac{1}{2-e^{-ix}}$$

$$\frac{1}{2-e^{ix}} = \frac{1}{2}\left(\frac{1}{1-q}\right)$$

$$q = \frac{1}{2}e^{ix}.$$

$$= \frac{1}{2}\left(1+q+q^2+\cdots\right)$$

$$= \frac{1}{2}+\frac{1}{4}e^{ix}+\frac{1}{8}e^{2ix}+\ldots$$

$$\frac{1}{2-e^{-ix}} = \frac{1}{2}+\frac{1}{4}e^{-ix}+\frac{1}{8}e^{-2ix}+\ldots$$

These geometric series have have exponentially fast decay as $\frac{1}{2^k}$.

$$\frac{1}{2-e^{ix}}+\frac{1}{2-e^{-ix}} = 1+\frac{\cos x}{2}+\frac{\cos 2x}{4}+\frac{\cos 3x}{8}+\ldots$$

$$\frac{1}{2-e^{ix}}+\frac{1}{2-e^{-ix}} = \frac{4-2\cos x}{5-4\cos x}$$

It defines the infinitely smooth function

Example 2 Find c_n for the 2π-periodic shifted pulse

$$F(x) = \begin{cases} 1, & \text{for } s < x < s+h \\ 0, & \text{elsewhere in } [-\pi, \pi]. \end{cases}$$

Solution

$$c_n = \frac{1}{2\pi} \int_s^{s+h} 1 \cdot e^{-inx}\, dx = \frac{1}{2\pi} \left(\frac{e^{-inx}}{-in} \right) \Bigg|_s^{s+h}$$

$$= \underbrace{e^{-ins}}_{\text{simple shift by } s} \left(\frac{1 - e^{-inh}}{2\pi i n} \right)$$

The energy is unchanged, the integral of $|F|^2$ just shifts and all

$$|e^{-ins}| = 1$$

Example 3 Centered pulse

Shift $\boxed{s = -\frac{h}{2}}$

The square pulse is centered around $x = 0$.

$$F(x) = \begin{cases} 1, & -\frac{h}{2} \le x \le \frac{h}{2} \\ 0, & \text{elswhere in } [-\pi, \pi]. \end{cases}$$

$$c_n = \frac{1}{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} 1 \cdot e^{-inx}\, dx = \frac{1}{2\pi} \left[\frac{e^{-inx}}{-in} \right]_{-\frac{h}{2}}^{h/2}$$

$$= \frac{1}{2\pi} \left(\frac{e^{-in\frac{h}{2}} - e^{in\frac{h}{2}}}{-in} \right)$$

$$= \frac{1}{2\pi} \frac{\sin(nh/2)}{n/2}$$

$$\boxed{\operatorname{sinc}(x) = \frac{\sin x}{x}}$$

sinc - funct

Let's denote

$$F_{centered}(x) := F(x)$$

$$\frac{F_{centered}}{h} = \begin{cases} 1/h & \text{for } -\frac{h}{2} \le x \le \frac{h}{2} \\ 0 & \text{elsewhere in } [-\pi, \pi] \end{cases}$$

$$\frac{F_{centered}}{h} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \underbrace{\operatorname{sinc}\left(\frac{nh}{2}\right)}_{c_n \text{ coefficient}} e^{inx}$$

We consider the limit $\boxed{h \to 0}$

$$\delta(x) = \begin{cases} 1, & x \neq 0 \\ \infty, & x = 0. \end{cases}$$

The Dirac delta function such that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

The theory of distributions

Converting between real and complex Fourier series

The Euler formula is the basic bridge:

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x,$$

$$\Rightarrow \quad \cos x = \frac{1}{2} e^{-ix} + \frac{1}{2} e^{ix}$$

$$\sin x = \frac{i}{2} e^{-ix} - \frac{i}{2} e^{ix}.$$

<u>Example</u> $F(x) = 5\cos x + 12 \sin x$

$$= 5 \times \left(\frac{1}{2} e^{-ix} + \frac{1}{2} e^{ix}\right) + 12 \times \left(\frac{i}{2} e^{-ix} - \frac{i}{2} e^{ix}\right) = \left(\frac{5}{2} + 6i\right) e^{-ix} + \left(\frac{5}{2} - 6i\right) e^{ix}$$

$$F(x) = (3+4i)\, e^{-2ix} + (3-4i)\, e^{2ix}$$

$$= \dots = 6\cos(2x) + 8\sin 2x$$

Some important properties of complex Fourier series

1. A scalar product of the vector space of complex valued functions with period 2π

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \overline{g(x)}\, dx$$

2. The norm:

$$(f, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \overline{f(x)}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx$$

We define the norm of f:

$$\| f \| = \sqrt{(f, f)}$$

Sometimes the second notation of the norm is used

$$| f | := \| f \|$$

The basis of Hilbert space

$$(e^{inx}, e^{inx}) = 1$$

$$(e^{inx}, e^{imx}) = 0 \quad \text{for } m \neq n.$$

The set

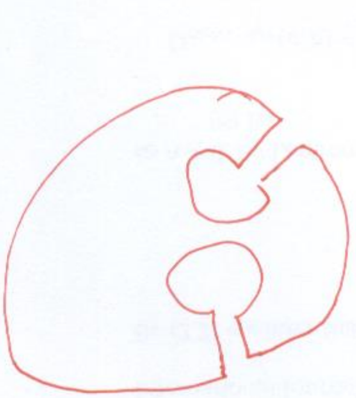
$\{e^{inx} \mid n \in \mathbb{Z}\}$ is an orthonormal set of basis vectors.

Evaluation of some integrals

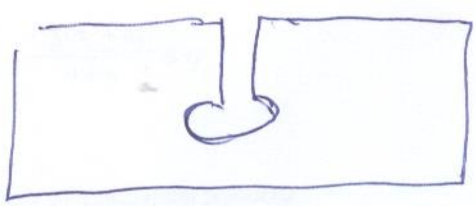
We are interested to see how some complicated real valued integrals can be evaluated by the use of Cauchy's theorem.

First we give some examples of toy contours

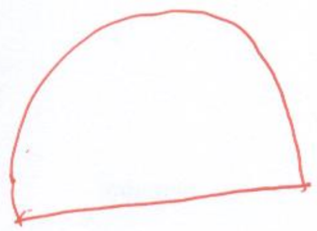
1. The multiple keyhole



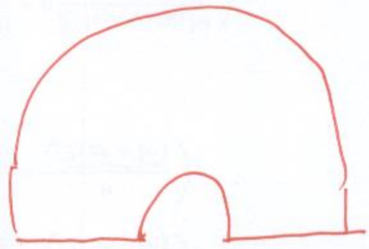
2. Rectangular keyhole



The semicircle



The indented semicircle



Example 1. Let consider the following integral

$\xi \in R$

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx = e^{-\pi \xi^2}$$

We see that $e^{-\pi x^2}$ is its own Fourier transform.

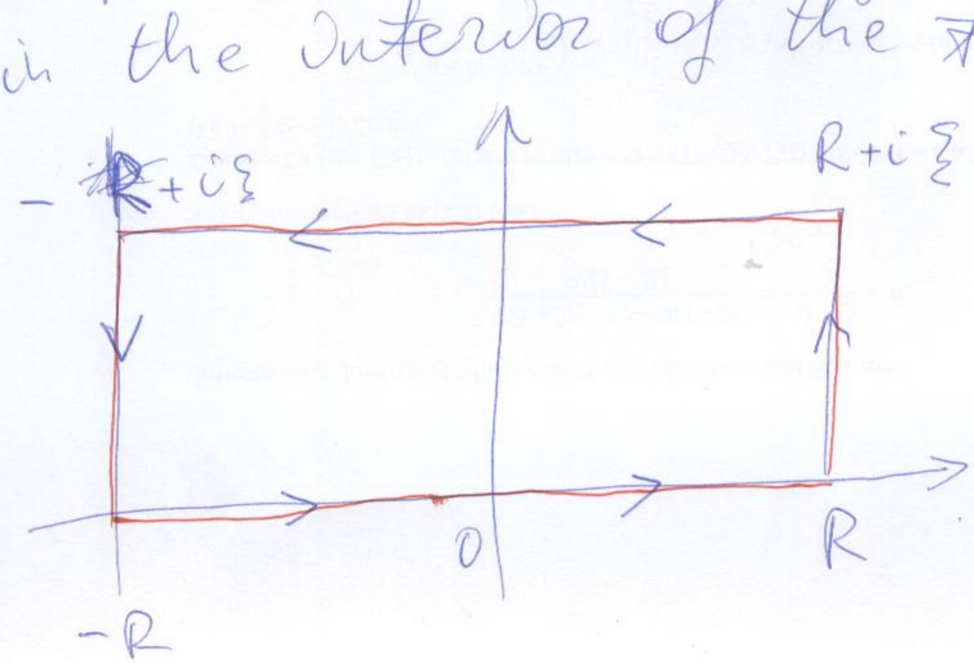
If we take $\xi = 0$, then we get the important result (this result is well known).

$$\boxed{\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1}$$

<u>Analysis</u>. Suppose that $\xi > 0$ and consider the function (complex valued!)

$$f(z) = e^{-\pi z^2}$$ (which is entire, has derivative $\forall z \in \mathbb{Z}$).

In particular it is analytic (holomorphic) in the interior of the ~~toy~~ contour γ_R



The contour γ_R consists of a rectangle with the positive counterclockwise orientation.

! By the Cauchy theorem

$$\boxed{\int_{\gamma_R} f(z)\, dz = 0.}$$

1. The integral over the real segment is simply

$$\int_{-R}^{R} e^{-\pi x^2}\, dx,$$

it converges to 1 as $R \to \infty$.

The integral on the vertical side on the right is

$$I_R(R) = \int_0^{\xi} f(R+iy)\, i\, dy = \int_0^{\xi} e^{-\pi(R^2+2iRy-y^2)}\, i\, dy$$

integral goes to

$\longrightarrow 0$ as $R \to \infty$. since ξ is fixed

and we may estimate it by

$$|I_R(R)| \leq C e^{-\pi R^2}.$$

Similary $I_L(R) \to 0$ as $R \to \infty$.

Integral over the horizontal segment on ~~to~~ is

$$\int_{-R}^{R} e^{-\pi(x+i\xi)^2}\, dx = -e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx$$

Therefore as $R \to \infty$

$$0 = 1 - e^{\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx$$

Example 2.

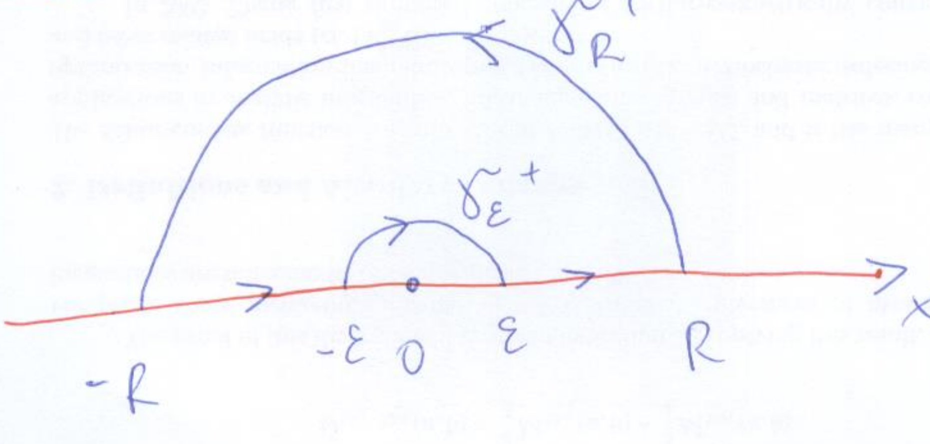
Show that

$$\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}.$$

We consider the function

$$f(z) = (1-e^{iz})/z^2$$

We integrate it over the indented semicircle



The Cauchy theorem gives

$$\int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz$$

$$+ \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_R^+} \frac{1-e^{iz}}{z^2} = 0.$$

We observe that

$$\left| \frac{1-e^{iz}}{z^2} \right| \leq \frac{2}{|z|^2}.$$

Let $R \to \infty$ so

$$\int_{\gamma_R^+} \frac{1-e^{iz}}{z^2} \xrightarrow[R\to\infty]{} 0.$$

Therefore

$$\int_{|x|\geq \varepsilon} \frac{1-e^{ix}}{x^2}\,dx = -\int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz$$

Note that

$$f(z) = -\frac{iz}{z^2} + E(z)$$

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \cdots$$

$$\frac{1-e^{iz}}{z^2} = -\frac{iz}{z^2} + \frac{1}{z^2}\left(\frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots\right)$$

$\rightarrow C$ as $z \to 0$

On γ_ε^+ we have

$z = \varepsilon e^{i\theta}$ and $dz = i\varepsilon e^{i\theta}\,d\theta$.

Thus :

$$\int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\, dz \xrightarrow[\varepsilon\to 0]{} \int_\pi^0 \frac{(-i)\cdot \varepsilon e^{i\theta}\, i\varepsilon e^{i\theta}}{\varepsilon^2 e^{2i\theta}}\, d\theta$$

$$= -\int_\pi^0 (-ii)\, d\theta = -\pi$$

Taking real parts we get

$$\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\, dx = \pi$$

and since the integral is even, then

$$\int_0^\infty \frac{1-\cos x}{x^2}\, dx = \frac{\pi}{2}$$