Lecture 9

Fourier integral and transform

First we introduce (without any strict justification) <u>Fourier integral</u>

Let's consider Fourier series in the complex form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i\pi n x}{l}}$$

with

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-\frac{i\pi n x}{l}} dx, \quad n = \dots, -2, -1, 0, 1, 2, \dots$$

($l = \pi$ can be taken)

We have Parseval's equality

$$2l \sum_{n=-\infty}^{\infty} |c_n|^2 = \int_{-l}^{l} |f(x)|^2 dx \qquad (*)$$

Next we introduce

$$\omega_n := \frac{\pi n}{\ell} \quad \text{and} \quad \Delta\omega_n = \omega_n - \omega_{n-1} = \frac{\pi}{\ell}$$

We rewrite Fourier series as

$$f(x) = \sum_{n=-\infty}^{\infty} C(\omega_n)\, e^{i\omega_n x}\, \Delta\omega_n$$

with

$$C(\omega_n) = \frac{1}{2\pi} \int_{-\ell}^{\ell} f(x)\, e^{-i\omega_n x}\, dx$$

where

$$C(\omega_n) = \frac{c_n}{\Delta\omega_n}.$$

(*) should be rewritten as

$$\int_{-\ell}^{\ell} |f(x)|^2 dx = 2\pi \sum_{n=-\infty}^{\infty} |C(\omega_n)|^2 \Delta\omega_n.$$

If we formally set $l \to \infty$, then $\Delta\omega_n \to 0$ and we get (take Riemannian sums)

$$f(x) = \int_{-\infty}^{\infty} C(\omega) e^{i\omega x} d\omega$$

with

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \qquad (*)$$

and

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 2\pi \int_{-\infty}^{\infty} |C(\omega)|^2 d\omega.$$

<u>Def</u>. Equality $(*)$ gives a Fourier transform of $f(x)$, denoted by

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

<u>Def</u>. Fourier integral defines also inverse Fourier transform

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

We have the following conservation equality

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 2\pi \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega.$$

Symmetrical version:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\hat{}} \hat{f}(\omega) e^{i\omega x} dx.$$

The given formulae can be rewritten as cos- and sin - Fourier transform and integral

$$f(x) = \int_0^\infty \Big(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x) \Big) d\omega$$

with (check it)

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(\omega x)\, dx$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(\omega x)\, dx$$

and

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \pi \int_0^\infty \Big(|A(\omega)|^2 + |B(\omega)|^2 \Big) d\omega$$

$A(\omega)$ and $B(\omega)$ are cos- and sin - Fourier transforms.

1. Case 1. $f(x)$ is even function

iff $B(\omega) = 0$.

2. Case 2. $f(x)$ is odd function

iff $A(\omega) = 0$.

Next we want to decompos each function on $[0, \infty)$ into cos-Fourier integral and sin-Fourier integral

(If possible.)

We can prolong the definition of $f(x)$ on $(-\infty, \infty)$ as:

a) an odd function

b) an even function.

<u>a case</u>: $f(x)$ is odd function

$$f(-x) = -f(x)$$

$$A(\omega) = 0 \qquad f(x) = \int_0^\infty B(\omega)\sin(\omega x)\,d\omega$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{0} f(x)\sin(\omega x)\,dx + \frac{1}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx$$

$$= \frac{1}{\pi}\int_0^\infty \overset{-f(x)}{\overset{\shortparallel}{f(-x)}}\ \overset{-\sin(\omega x)}{\sin(\omega(-x))}\,d(+x) + \frac{1}{\pi}\int_0^\infty f(x)\sin\omega x\,dx$$

$$= \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx$$

b case

$$f(-x) = f(x), \quad B(\omega) = 0.$$

$$f(x) = \int_0^\infty A(\omega)\cos(\omega x)\,d\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\omega x)\,dx.$$

Recall:

Th. Let f be a piecewise continuously differentiable function. Then the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^\infty \left(a_n \cos\left(\frac{\pi n x}{l}\right) + b_n \sin\left(\frac{\pi n x}{l}\right)\right)$$

$$a_n = \frac{1}{l}\int_J f(x)\cos\left(\frac{\pi n x}{l}\right)dx, \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{l}\int_J f(x)\sin\left(\frac{\pi n x}{l}\right)dx \quad n = 1, 2, \ldots$$

$$J := [x_0, x_1], \quad x_1 - x_0 = 2l.$$

converges to

a) $f(x)$ if f is continuous at x

b) $\frac{1}{2}\left(f(x+0)+f(x-0)\right)$ if f is discontinuous at x.

(x is an internal point)

The same statement holds for Fourier Integral (in the real form)

$$\int_0^\infty \left(A(\omega)\cos(\omega x)+B(\omega)\sin(\omega x)\right)d\omega$$

Complex case: convergence means $(-\infty,\infty)$ interval

$$\lim_{M,N\to\infty}\sum_{n=-M}^{N} c_n e^{\frac{inx}{\ell}}$$

$$\lim_{M,N\to\infty}\int_{-M}^{N} C(\omega)e^{i\omega x}\,d\omega$$

M,N tend to ∞ independently.