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GRID APPROXIMATION OF NONSMOOTH SOLUTIONS OF SINGULARLY PERTURBED EQUATIONS

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Abstract. We consider the Dirichlet problem for a singularly perturbed reaction-diffusion equation on the unit square. It is assumed that the coefficients of the equation, its right-hand side and the boundary conditions on the sides of the square are enough smooth functions. Any compatibility conditions in the corner points are not assumed to be satisfied.

We introduce the piecewise uniform condensing Shishkin mesh in the domain. For the numerical solution of the problem under consideration, we use a classical five-point difference scheme on this mesh. We prove that the approximate solution ε-uniformly converges at the rate $O(N^{-2+\delta})$ in the L^h_∞-norm, where N is the number of grid nodes in one direction and $\delta > 0$ is arbitrary.

For the equation under consideration the best estimate $O(N^{-2})$ is proved only under assumptions on more high smoothness of the desired solution, namely $u \in C^{4+\lambda}(\overline{\Omega})$ when additional compatibility conditions are satisfied. For the case of a smoothness we use, the best of known to the author estimate is $O(N^{-1/4})$.

Key words: singular perturbation, nonsmooth solution, uniform convergence

1. Introduction

We consider the Dirichlet problem for a singularly perturbed reaction-diffusion equation

$$-\varepsilon^2 \Delta u + q(x,y)u = f(x,y), \quad (x,y) \in \Omega, \tag{1.1}$$

$$u = g(x,y), \quad (x,y) \in \partial\Omega \equiv \Gamma = \bigcup_{k=1}^{4} \Gamma_k, \tag{1.2}$$

where $\varepsilon \in (0,1]$ is a small parameter, Ω is a unit square and $\Gamma_k = \overline{\Gamma}_k$ are its sides enumerated counter-clockwise from $\Gamma_1 = \{(x,y) \in \Gamma | x = 0\}$. The vertices of the square are enumerated in the same way from the vertex, which is in the origin of coordinates. We suppose the coefficient $q(x, y)$ satisfies the following condition

$$q(x, y) > \alpha^2 = \text{const } > 0. \tag{1.3}$$

It is well known that for a small ε the solution of problem (1.1)-(1.3) has the boundary layer along the whole boundary Γ. This causes certain difficulties in numerical solving of this problem. Also it is known that presence of corner points on the boundary of the domain (in our case they are the vertices of the square) unfavorably affects on smoothness of the exact solution and therefore makes worse an accuracy of the approximate solution.

Assume that the coefficient of equation (1.1) and its right-hand side belong to the Hölder class $C^{4,\lambda},\ 0 < \lambda < 1$ on the closure $\overline{\Omega}$ of the domain Ω

$$q(x, y),\ f(x, y) \in C^{4,\lambda}(\overline{\Omega}). \tag{1.4}$$

Assume as well that the restriction of the boundary function $g(x, y)$ to the sides of the square has appropriate smoothness

$$g_k(s) := g(x, y)\big|_{\overline{\Gamma}_k} \in C^{4,\lambda}(\overline{\Gamma}_k), \quad k = 1, 2, 3, 4. \tag{1.5}$$

If it is true that

$$g(x, y) \in C(\partial\Omega), \tag{1.6}$$

then it follows from (1.4), (1.5) that

$$u(x, y) \in C^{1,\lambda'}(\overline{\Omega}) \cap C^{4,\lambda}(\Omega), \tag{1.7}$$

where $\lambda' \in (0, 1)$ is an arbitrary number.

In order that the solution of problem (1.1)-(1.5) to be more smooth in $\overline{\Omega}$ than (1.7), it is necessary to impose so called compatibility conditions on the coefficients and right-hand sides of the equation and the boundary conditions at the corner points of the boundary.

In this work input data of problem (1.1)-(1.3) are assumed to satisfy conditions (1.4)-(1.6) only. *Any additional compatibility conditions at the corner points are not assumed to be satisfied.*

There is a wealth of literature on numerical methods for problem (1.1)-(1.2). We shall concern methods, uniformly convergent with respect to ε in L^h_∞-norm. We do not concern fitting schemes and one-dimensional problems.

In 1987 Shishkin [5] has considered problem (1.1)-(1.3) on the mesh of Bakhvalov's type [1] and, under assumption $u \in C^{4,\lambda}(\overline{\Omega})$, proved the estimate $|u^h_{ij} - u(x_i, y_j)| = O(N^{-2})$. For smoothness (1.7), in the same work the estimate $O(N^{-2/11})$ was proved.

In the book [6] Shishkin made use of piecewise uniform mesh (the Shishkin mesh) for solving problem (1.1)-(1.3) and for sufficient smoothness of the solution he obtained the estimate $O(N^{-1} \ln N)$, and also, when the compatibility conditions are absent, proved the estimate $O((N^{-1} \ln N)^{1/4})$.

In 2003 Shishkin [7] has considered problem (1.1)-(1.3) for $u \in C^{4,\lambda}(\overline{\Omega})$ on the piecewise uniform mesh with several points of change of mesh size and obtained the estimate $O(N^{-2}(\ln \ln \ldots \ln N))^2$.

In 2005 Clavero, Gracia and O'Riordan [2], for $u \in C^{4,\lambda}(\overline{\Omega})$ on Shishkin mesh, have obtained estimate $O(N^{-2}\ln^2 N)$.

It is impossible not to mention that Bakhvalov (1969), in [1] for rather different from (1.1) equation without compatibility conditions, has constructed the approximate solution on his mesh, with accuracy of $O(N^{-2})$.

2. Known Results

The following theorem takes place.

Theorem 1 [Volkov [8], Han-Kellogg [3]]. *If conditions* (1.4)–(1.6) *are fulfilled, then the solution of problem* (1.1)-(1.3) *admits the decomposition of the form*

$$u(x,y)=\sum_{l=1}^{4}\sum_{k=1}^{2}a_k^l\varphi_k^l(x,y)+w(x,y),$$

where $w(x,y)\in C^{4,\lambda}(\overline{\Omega})$*, and, for example,*

$$\varphi_k^1(x,y)=Im\,(\zeta^{2k}\ln\zeta)\ \ for\ \ \varepsilon\zeta=x+i\,y.$$

Vanishing of the coefficients a_k^l means fulfillment of the appropriate compatibility conditions. In particular, it follows from Theorem 1 that, for example, in the neighbourhood of the vertex 1(the origin of coordinates)

$$\left|D_{xy}^4u\right|\leqslant c\,(\varepsilon r)^{-2},\quad r=\sqrt{x^2+y^2}. \tag{2.1}$$

Theorem 2 [Clavero-Gracia-O'Riordan [2]]. *If conditions* (1.4)–(1.6) *are fulfilled, then the solution of problem* (1.1)-(1.3) *admits the following decomposition*

$$u(x,y)=U(x,y)+\sum_{k=1}^{4}w_k(x,y)+\sum_{k=1}^{4}v_k(x,y), \tag{2.2}$$

where

$$\mathcal{L}U=f,\quad \mathcal{L}w_k=0,\quad \mathcal{L}v_k=0,\quad k=1,2,3,4,$$

besides, $U(x,y)\in C^{4,\lambda}(\overline{\Omega})$ *is a regular term of the solution,* $w_k\in C^{4,\lambda}(\overline{\Omega})$ *are exponential boundary layer functions,*

$$v_k\in C^{1,\lambda'}(\overline{\Omega})\cap C^{4,\lambda}(\overline{\Omega}\setminus\ neighborhood\ of\ the\ vertex\ k)$$

are corner layer functions.

Let us introduce the piecewise uniform Shishkin mesh in the domain $\overline{\Omega}$

$$\overline{\Omega}^h=\overline{\omega}(x_i)\times\overline{\omega}(y_j), \tag{2.3}$$

where $\overline{\omega}$ is a one-dimensional piecewise uniform mesh being condensed to the ends of the segment $[0,1]$, which is defined as follows

$$\overline{\omega}(s_i) = \{s_i = \xi(i/N), \quad i = 0, 1, \dots, N\}, \tag{2.4}$$

and

$$\xi(t) = \begin{cases} 4\sigma t, & 0 \leqslant t \leqslant 1/4, \\ \sigma + 2(1-2\sigma)(t-1/4), & 1/4 \leqslant t \leqslant 3/4, \\ 1 - 4\sigma(1-t), & 3/4 \leqslant t \leqslant 1, \end{cases} \tag{2.5}$$

$$\sigma = \min\left\{\frac{1}{4}, \frac{2\varepsilon}{\alpha}\ln N\right\}.$$

On the mesh $\overline{\Omega}^h$ we approximate problem (1.1)-(1.3) by a classical difference scheme

$$\begin{aligned} \mathcal{L}^h u^h := -\varepsilon^2(u^h_{\bar{x}\hat{x}} + u^h_{\bar{y}\hat{y}})_{ij} + q_{ij}u^h_{ij} = f_{ij}, \quad (x_i, y_j) \in \Omega^h := \overline{\Omega}^h \cap \Omega, \\ u^h_{ij} = g(x_i, y_j), \quad (x_i, y_j) \in \Gamma^h := \Gamma \cap \overline{\Omega}^h. \end{aligned} \tag{2.6}$$

The following representation takes place for the solution of problem (2.6)

$$u^h_{ij} = U^h_{ij} + \sum_{k=1}^{4} w^h_{k,ij} + \sum_{k=1}^{4} v^h_{k,ij}, \quad (x_i, y_j) \in \overline{\Omega}^h, \tag{2.7}$$

where

$$\mathcal{L}^h U^h_{ij} = f_{ij}, \quad \mathcal{L}^h w^h_{k,ij} = \mathcal{L}^h v^h_{k,ij} = 0, \quad k = 1, 2, 3, 4,$$

and the functions U^h_{ij}, $w^h_{k,ij}$, $v^h_{k,ij}$ are the grid approximations of the functions $U(x, y)$, $w_k(x, y)$, $v_k(x, y)$ from representation (2.2). It follows from [2] that

$$\begin{aligned} \|U^h_{ij} - U(x_i, y_j)\|_{L^h_\infty(\overline{\Omega}^h)} &:= \max_{(x_i, y_j) \in \overline{\Omega}^h} |U^h_{ij} - U(x_i, y_j)| = O(N^{-2}\ln^2 N), \\ \|w^h_{k,ij} - w_k(x_i, y_j)\|_{L^h_\infty(\overline{\Omega}^h)} &= O(N^{-2}\ln^2 N), \quad k = 1, 2, 3, 4. \end{aligned} \tag{2.8}$$

Let $\overline{\Omega}^h_1 = \overline{\Omega}^h \cap \{(x, y)\, 0 \leqslant x, y \leqslant \sigma\}$ be a subset of the mesh points of $\overline{\Omega}^h$, which forms a square mesh with a small mesh size h in the neighborhood of the vertex 1 and Ω^h_1 be its inner part. Let $\overline{\Omega}^h_k$ denote similar subsets from the neighborhoods of the vertices k. It follows from [2] that

$$\|v^h_{k,ij} - v_k(x_i, y_j)\|_{L^h_\infty(\overline{\Omega}^h \setminus \Omega^h_k)} = O(N^{-2}), \quad k = 1, \dots, 4. \tag{2.9}$$

3. The Main Theorem

Theorem 3 [main]. *If conditions* (1.4)–(1.6) *are satisfied then the solution of problem* (2.6) *on the mesh* (2.3)-(2.5) *converges uniformly with respect to* ε *to the solution of problem* (1.1)-(1.3) *at the rate of* $O(N^{-2+\delta})$ *in sense of the* L^h_∞*-norm, where* $\delta > 0$ *is an arbitrary number, that is*

$$\|u^h_{ij} - u(x_i, y_j)\|_{L^h_\infty(\overline{\Omega}^h)} = O(N^{-2+\delta}). \tag{3.1}$$

By virtue of representations (2.2),(2.7) and estimates (2.8),(2.9), for validity of (3.1) it is sufficient to prove that

$$\|v^h_{k,ij} - v_k(x_i, y_j)\|_{L^h_\infty(\Omega^h_k)} = O(N^{-2+\delta}), \quad k = 1, \dots, 4.$$

Actually it is sufficient to carry out this estimate only for $k = 1$, because for the other k this is made similarly. The function $z^h_{ij} := v^h_{1,ij} - v_1(x_i, y_j)$ is the solution of the following problem

$$\mathcal{L}^h z^h_{ij} = \psi_{ij} := -\mathcal{L}^h v_1(x_i, y_j), \quad (x_i, y_j) \in \Omega^h_1, \qquad z^h_{ij}\,\big|_{\partial\Omega^h_1} \text{ is defined.} \tag{3.2}$$

Theorem 4. *For the solution of problem* (3.2) *the following a priori estimate is valid*

$$\|z^h\|_{L^h_\infty(\Omega^h_1)} \leqslant c\,\varepsilon^{-2} \ln N \|\psi\|_{L^h_1(\Omega^h_1)} + \|z^h\|_{L^h_\infty(\partial\Omega^h_1)}, \tag{3.3}$$

where

$$\|v\|_{L^h_1(\Omega^h_1)} := (|v|, 1)_{\Omega^h_1} := \sum_{(x_i, y_j) \in \Omega^h_1} |v_{ij}| h^2.$$

Since by virtue of (2.1) $|\psi_{ij}| \leqslant c\,h^2/(x_i^2 + y_i^2)$, and from (2.9) it follows that $\|z^h\|_{L^h_\infty(\partial\Omega^h_1)} = O(N^{-2})$, then from (3.3) we obtain desired estimate

$$\|z^h\|_{L^h_\infty(\Omega^h_1)} \leqslant c\,\varepsilon^{-2} h^2 \ln^2 N = O(N^{-2} \ln^4 N) = O(N^{-2+\delta}).$$

In order to prove theorem 4 we represent the solution of problem (3.2) using the Green function, which is defined, for every $Q = (\xi'_i, \eta'_j)$ by the relations

$$\mathcal{L}^h G(P; Q) = \begin{cases} 0, & P \neq Q, \\ h^{-2}, & P = Q, \end{cases} \quad P = (x_i, y_j) \in \Omega^h_1,$$
$$G(P; Q) = 0, \quad P \in \partial\Omega^h_1.$$

For the Green function the following relations, easily verified, hold

$$\varepsilon^2 \|\nabla^h G\|^2_{L^h_2(\overline{\Omega}^h_1)} + (q, G^2)_{\Omega^h_1} = G(P, P), \tag{3.4}$$

$$\varepsilon^2 /G_n, 1/ + (q, G)_{\Omega^h_1} = 1, \tag{3.5}$$

where $\nabla^h G$ is a grid analogue of the gradient and $/G_n, 1/$ is a discrete analogue of the integral along the boundary from the derivative with respect to the inner normal.

By virtue of (1.3) and positiveness of the Green function, it follows from (3.5) that $G_n \geqslant 0$ and

$$\varepsilon^2 /G_n, 1/ \leqslant 1, \tag{3.6}$$

and also, from (3.4), we get the following estimate

$$\varepsilon^2 \|\nabla^h G\|^2_{L^h_2(\overline{\Omega}^h_1)} \leqslant G(P; P). \tag{3.7}$$

Using (3.4), it is easy to verify that $\|\nabla^h G\|^2_{L_2^h(\Omega_1^h)}$ does not depend on the value of the mesh size h of $\overline{\Omega}_1^h$. Therefore so called "weak embedding theorem" [4] takes place, in virtue of which, for any function v_{ij} defined on $\overline{\Omega}_1^h$ and vanishing on $\partial\Omega_1^h$, it follows that

$$\|v_{ij}\|_{L_\infty^h(\Omega_1^h)} \leqslant c\,(\ln N)^{1/2}\|\nabla^h v\|_{L_2^h(\overline{\Omega}_1^h)}, \tag{3.8}$$

where constant c is independent of N. Therefore it follows from (3.7) that

$$\varepsilon^2\|\nabla^h G\|_{L_2^h(\overline{\Omega}_1^h)} \leqslant c\,(\ln N)^{1/2},$$

and from this and (3.8) we get

$$\|G\|_{L_\infty^h(\Omega_1^h\times\Omega_1^h)} \leqslant c\,\varepsilon^{-2}\ln N. \tag{3.9}$$

Using grid Green's formulas, we obtain

$$z^h(P) = \sum_{Q\in\Omega_1^h} G(P;Q)\psi(Q)h^2 + \varepsilon^2/G_n, z^h/$$

and therefore the following estimate is valid

$$\|z^h\|_{L_\infty^h(\Omega_1^h)} \leqslant \|G\|_{L_\infty^h(\Omega_1^h\times\Omega_1^h)}\|\psi\|_{L_1^h(\Omega_1^h)} + \varepsilon^2/G_n, 1/\ \|z^h\|_{L_\infty^h(\partial\Omega_1^h)}.$$

Combining this estimate with (3.6) and (3.9), we obtain a priori estimate (3.3).

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