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GENERAL CONVOLUTIONS OF INTEGRAL TRANSFORMS AND THEIR APPLICATION TO ODE AND PDE PROBLEMS

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Abstract. The present research is devoted to some polyconvolutions generated by various integral transforms. For example, we study convolutions of Hankel transform with the following factorization properties:

$$\mathrm{H}_\nu[h](x) = \mathrm{H}_\mu[f](x)\mathrm{H}_{\mu+\nu}[g](x), \quad \mathrm{H}_{\nu+m}[h](x) = x^{-\nu}\mathrm{H}_{\nu+m}[f](x)\mathrm{H}_\nu[g](x),$$

$$\mathrm{H}_\nu[h](x) = x^{-\nu}\mathrm{H}_\mu[f](x)\mathrm{H}_\mu[g](x), \quad \mathrm{H}_\mu[h](x) = x^{-\nu}\mathrm{H}_\nu[f](x)\mathrm{H}_\mu[g](x),$$

where $\mathrm{H}_\nu[f](x)$ is the Hankel transform. Conditions for the existence of the constructed polyconvolutions are found. The results of this research are applied for solvability of ODE and PDE by the method of integral transforms. The derived constructions allow us to solve various nonuniform equations.

Key words: integral transforms, convolution, Hankel transform, ODE, PDE

1. Introduction

The definition of polyconvolution, or generalized convolution, was first introduced by V.A. Kakichev in 1967 [5]. This is the key definition of the present research.

Definition 1. Let A_1, A_2 and A_3 be operators. The generalized convolution of function $f(t)$ and $k(t)$, under A_1, A_2, A_3, with weighted function $\alpha(x)$, is the function $h(t)$ denoted by $\left(f_{A_1} \overset{\alpha}{*} k_{A_2}\right)_{A_3}(t)$ for which the following factorization property is valid:

$$(A_3 h)(x) = A_3\left[\left(f_{A_1} \overset{\alpha}{*} k_{A_2}\right)_{A_3}\right](x) = \alpha(x)(A_1 f)(x)(A_2 k)(x).$$

With the help of this definition we can construct polyconvolutions generated by various integral transforms. This paper is devoted to the construction and study of generalized convolutions generated by the Hankel transform.

The Hankel transform is the most extensively studied area of the theory of Bessel transforms. This transform is used to solve many problems of mathematical physics. It is defined by the integral

$$F_\nu(u) = \mathrm{H}_\nu[f](u) = \int_0^\infty f(t) t J_\nu(ut)\, dt, \quad x \in \mathbf{R}_+ , \tag{1.1}$$

where $J_\nu(z)$ is the Bessel function of the first kind of order ν, $\operatorname{Re}\nu > -1/2$.

The classical convolution of the Hankel transform was first introduced by Ya.I. Zhitomirskii in 1955 [11]. In 1967 V.A. Kakichev constructed this convolution with help of definition [5].

The explicit expression of this convolution is given by

$$\left(f_\nu \overset{-\nu}{*} k_\nu\right)_\nu (t) = \frac{t^\nu}{2^\nu \sqrt{\pi}\,\Gamma(\nu+1/2)} \int_0^\pi \sin^{2\nu} s\, ds \int_0^\infty f(\tau) \times \frac{g\left(\sqrt{t^2+\tau^2-2t\tau\cos s}\right)}{\left(t^2+\tau^2-2t\tau\cos s\right)^{\nu/2}}\, \tau^{\nu+1}\, d\tau . \tag{1.2}$$

The Hankel transform is the Mellin convolution type transform. There are many works, which are devoted to study of these transforms and their convolutions (see, for example, [4, 9, 10]). A number of convolution constructions involving the Hankel transform was derived by N.X.Thao and N.T.Xai [8]. Some polyconvolutions obtained by the author with the help of an approach by V.A. Kakichev were given in [6].

2. The Generalized Convolutions of Hankel Transform

The polyconvolutions with the factorization properties

$$\mathrm{H}_\nu[h_1](x) = x^{-\nu}\mathrm{H}_\mu[f](x)\mathrm{H}_\mu[g](x), \tag{2.1}$$

$$\mathrm{H}_\mu[h_2](x) = x^{-\nu}\mathrm{H}_\nu[f](x)\mathrm{H}_\mu[g](x). \tag{2.2}$$

were introduced in [2]. Consider the functions

$$h_1(t) = t^{-\nu} \iint\limits_{|u-v|<t<u+v} u^\nu f(u) g(v) P_1(t;u,v)\, du\, dv - t^{-\nu} \int_0^\infty du \int_0^{t-u} u^\nu f(u) g(v) Q_1(t;u,v)\, dv\, , \tag{2.3}$$

$$h_2(t) = t^{\nu-1} \int_0^\infty dv \int_{|t-v|}^{t+v} u^{-\nu+1} f(u) g(v) P_2(t;u,v)\, du$$

$$-t^{\nu-1} \int_0^\infty dv \int_{t+v}^{\infty} u^{-\nu+1} f(u) g(v) Q_2(t;u,v)\, du\,, \qquad (2.4)$$

where

$$P_i(t;u,v) = \frac{1}{\sqrt{2\pi}} v^\nu\, P^{1/2-\nu}_{\mu-1/2}(\cos s_i) \sin^{\nu-1/2} s_i\,,$$

$$Q_i(t;u,v) = \frac{\sqrt{2}}{\pi^{3/2}} \sin\Big[(\mu-\nu)\pi\Big] \mathbf{e}^{\frac{(2\nu-1)\pi}{2i}} v^\nu\, Q^{1/2-\nu}_{\mu-1/2}(\mathrm{ch} r_i) \mathrm{sh}^{\nu-1/2} r_i\,,\ i = 1, 2\,,$$

$P^\nu_\mu(x)$, $Q^\nu_\mu(x)$ are the associated Legendre functions of the first and second kind, respectively, and

$$2uv \cos s_1 = u^2 + v^2 - t^2\,,\ 2uv\, \mathrm{ch}\, r_1 = t^2 - u^2 - v^2\,,$$
$$2tv \cos s_2 = t^2 + v^2 - u^2\,,\ 2tv\, \mathrm{ch}\, r_2 = u^2 - t^2 - v^2\,.$$

The following theorems can be proved [2].

Theorem 1. *Suppose that* $\sqrt{t} f(t)$, $\sqrt{t} g(t) \in L(0,\infty)$ *and* $\mathrm{Re}\,\nu > 1/2$, $\mathrm{Re}\,\mu > (2\mathrm{Re}\,\nu - 3)/4$. *Then the function* $h_1(t)$ *exists and the factorization relation (2.1) is valid.*

Theorem 2. *Suppose that* $\sqrt{t} f(t)$, $\sqrt{t} g(t) \in L(0,\infty)$ *and* $\mathrm{Re}\,\nu > 1/2$, $\mathrm{Re}\,\mu > \mathrm{Re}\,\nu - 1$. *Then the function* $h_2(t)$ *exists and the factorization relation (2.2) is valid.*

These theorems present the conditions of existence of polyconvolutions (2.3) and (2.4).

The other convolution construction generated by the Hankel transform is the polyconvolution with the factorization property

$$\mathrm{H}_{\nu+m}[h](x) = x^{-\nu} \mathrm{H}_{\nu+m}[f](x) \mathrm{H}_\nu[g](x). \qquad (2.5)$$

In this case we prove the following theorem [3].

Theorem 3. *Suppose that* $\sqrt{t} f(t)$, $\sqrt{t} g(t) \in L(0,\infty)$, $g(t)$ *is the continuous function with a bounded variation on the any interval* $(0, R)$ *and* $\mathrm{Re}\,\nu > 1/2$. *Then the polyconvolution*

$$h(t) = \frac{2^{\nu-1} t^\nu m! \Gamma(\nu)}{\pi \Gamma(2\nu + m)} \int_0^\pi C^\nu_m(\cos s) \sin^{2\nu}\, ds$$

$$\times \int_0^\infty f(\tau) \frac{g(\sqrt{t^2+\tau^2-2t\tau\cos s})}{(t^2+\tau^2-2t\tau\cos s)^{\frac{\nu}{2}}} \tau^{\nu+1}\, d\tau\,, \qquad (2.6)$$

exists and the factorization relation (2.5) is valid.

Sometimes we need to introduce the differential operators [1]

$$N_{m,\nu} = t^{\nu} \left(\frac{d}{t\,dt} \right)^{m} t^{m-\nu}\,, \quad S^{k}_{m,\nu} = \left[N_{m,-\nu} N_{m,\nu+m} \right]^{k}\,, \tag{2.7}$$

which allow us to find the conditions of existence of polyconvolutions. These operators possess the following properties:

a) $S^{k}_{m,\nu} = S^{k}_{m,-\nu} = S_{km,\nu}\,,$
where $S_{n,\nu} = S^{1}_{n,\nu} = S^{n}_{1,\nu} = \left[\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{\nu^2}{t^2} \right]^{n}$.

b) $N_{m,\,\pm\nu+km} N_{m,\,\pm\nu+(k-1)m} ... N_{m,\,\pm\nu+m} N_{m,\,\pm\nu} = N_{m(k+1),\,\pm\nu+km}$.

It should be noted that some special cases of these operators occur in the main equations of mathematical physics, for example, in elasticity theory.

Using properties (a) and (b), it is readily verified that all well-know differential operators related to Hankel transform can be expressed in terms of $N_{m,\,\pm\nu}$ and $S_{m,\nu}$. Therefore, we restrict ourselves to the operators (2.7). Suppose that $C_{\lambda}(0,\infty)_{\rho}$ is the set of functions $f(t)$ with continuous derivatives on the interval $(0,\infty)$ such that the asymptotic estimates

$$f(t) = O\left(t^{\lambda}\right),\ t \to +0\,,\quad f(t) = O\left(t^{-\rho}\right),\ t \to +\infty$$

hold [1]. Further, $\rho, \lambda \in \mathbf{R}$, $\rho > 3/2$ and $\lambda > -1/2$. In this case we can construct, for example, the polyconvolution with the factorization property

$$\mathrm{H}_{\nu}[h](x) = \mathrm{H}_{\mu}[f](x)\mathrm{H}_{\mu+\nu}[g](x). \tag{2.8}$$

Conditions for the existence of this polyconvolution are not published somewhere before.

Theorem 4. *Suppose that* $\sqrt{t}f(t) \in L(0,\,\infty)$, $\sqrt{t}N_{2,\pm(\mu+\nu)+2}g(t) \in L(0,\,\infty)$, $N_{k,\pm(\mu+\nu)+k}g(t) \in C_{\lambda}(0,\infty)_{\rho}$, $k = 0,1$, $\mathrm{Re}\,\mu > -\frac{1}{2}$, $\mathrm{Re}\,(\mu+\nu) > -\frac{1}{2} + \max\{0, \mp 2\}$*, then the polyconvolution*

$$h(t) = \frac{2^{\frac{\mu+\nu}{2}} t^{\nu}}{\pi} \int\limits_{-\pi/2}^{\pi/2} \mathrm{e}^{\,i(\mu-\nu)s} \cos^{\frac{\mu+\nu}{2}} s$$
$$\times \int\limits_{0}^{\infty} f(\tau) g\left(\sqrt{2(\tau^2 \mathrm{e}^{\,is} + t^2 \mathrm{e}^{\,-is}) \cos s} \right) (\tau^2 \mathrm{e}^{\,is} + t^2 \mathrm{e}^{\,-is})^{-\frac{\mu+\nu}{2}} \tau^{\mu+1}\, d\tau\,. \tag{2.9}$$

exists and the factorization relation (2.8) is valid.

3. Application of Polyconvolution to ODE and PDE

The results of this research are applied for solvability of ODE and PDE by the method of integral transforms [7]. The derived constructions allow us to solve various nonuniform equations.

Consider the equation

$$\sum_k a_k \mathcal{L}^k u(t) = f(t)\,, \tag{3.1}$$

where $\mathcal{L}^k$ is a operator. For example, $\mathcal{L}^k u(t) = \frac{d^k u(t)}{dt^k}$ or $\mathcal{L}^k u(t) = u(t-\omega_k)$. The solution of this equation can be represented as a polyconvolution

$$u(t) = \left(\varphi_{B_1} \overset{\alpha}{*} \psi_{B_2}\right)_A (t)\,. \tag{3.2}$$

Example 1. Consider the equation (3.1) with the differential operators (2.7)

$$\mathcal{L}^k h(t) = S_{k,\nu} h(t), \quad k = 0,1,2,\dots n, \quad t \in \mathbf{R}_+\,,$$

and zero conditions. Then

$$\psi(t) = \mathrm{H}_\nu^{-1}[\Psi(x)]\,, \quad \Psi(x) = \frac{x^\nu}{\sum\limits_{k=0}^{n} (-1)^k a_k x^{2k}}$$

and the solution of this equation is given by

$$u(t) = \left(f_\nu \overset{-\nu}{*} \psi_\nu\right)_\nu (t).$$

If $n = 1\,;\ a_0 = a\,,\ a_1 \equiv 1\,;\ \dfrac{1}{2} < \operatorname{Re}\nu < \dfrac{3}{2}$ we have the following equation

$$\frac{d^2u}{dt^2} + \frac{du}{t\,dt} - \left(\frac{\nu^2}{t^2} - a\right) u(t) = f(t)$$

and we obtain

$$\psi(t) = \frac{\pi}{2}|c|^\nu Y_\nu(|c|t),\ \text{if } a = c^2, \quad \psi(t) = -|c|^\nu K_\nu(|c|t),\ \text{if } a = -c^2\,,$$

where $Y_\nu(z)$ and $K_\nu(z)$ are the Bessel function and the modified Bessel function of the second kind of order ν.

Example 2. Let us consider the equation

$$\frac{\partial^2 v}{\partial {t_1}^2} - b^2\left(\frac{\partial^2 v}{\partial {t_2}^2} + \frac{1}{t_2}\frac{\partial v}{\partial t_2}\right) + \left(\gamma + \frac{b^2\nu^2}{{t_2}^2}\right) v(t_1,t_2) = f(t_1,t_2)$$

with the conditions $u(+0,t_2) = u_0(t_2),\ u'_{t_1}(+0,t_2) = u_1(t_2)$. Then the solution of this equation is given by

$$\begin{aligned} u(t_1,t_2) &= (f*\psi)_{\mathrm{L},\nu}(t_1,t_2) + (u_0 * \psi_{t_1})_\nu(t_1,t_2) + (u_1*\psi)_\nu(t_1,t_2) \\ &= (f*\psi)_{\mathrm{L},\nu}(t_1,t_2) + \frac{\partial}{\partial t_1}(u_0*\psi)_\nu(t_1,t_2) + (u_1*\psi)_\nu(t_1,t_2), \end{aligned}$$

where $\psi(t_1,t_2) = \mathrm{L}^{-1}\mathrm{H}_\nu^{-1}\left[\dfrac{\xi^\nu}{p^2 + b^2\xi^2 + c}\right]$,

$$(f*\psi)_{\mathrm{L},\nu}(t_1,t_2)=\frac{t_2^{\nu}}{2^{\nu}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\int\limits_0^{t_1}d\xi\int\limits_0^{\pi}\sin^{2\nu}s\,ds\int\limits_0^{\infty}f(\xi,\zeta)$$

$$\times\frac{\psi(t_1-\xi,\sqrt{t_2^2+\zeta^2-2t_2\zeta\cos s})}{(t_2^2+\zeta^2-2t_2\zeta\cos s)^{\frac{\nu}{2}}}\,\zeta^{\nu+1}\,d\zeta,$$

$$(u_k*\psi)_{\nu}(t_1,t_2)=\frac{t_2^{\nu}}{2^{\nu}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\int\limits_0^{\pi}\sin^{2\nu}s\,ds\int\limits_0^{\infty}u_k(\zeta)$$

$$\times\frac{\psi(t_1,\sqrt{t_2^2+\zeta^2-2t_2\zeta\cos s})}{(t_2^2+\zeta^2-2t_2\zeta\cos s)^{\frac{\nu}{2}}}\,\zeta^{\nu+1}\,d\zeta,\quad k=0,1.$$

If $c=\gamma^2>0$, $\gamma\in\mathbf{R}$, then

$$\psi(t_1,t_2)=\frac{|\gamma|^{\nu+\frac{1}{2}}t_2^{\nu}}{\sqrt{2\pi}|b|^{\nu+\frac{3}{2}}}(b^2t_1^2-t_2^2)^{-\frac{\nu}{2}-\frac{1}{4}}\eta(b^2t_1^2-t_2^2)J_{-\nu-\frac{1}{2}}\left(\frac{|\gamma|}{|b|}\sqrt{b^2t_1^2-t_2^2}\right).$$

If $c=0$, then $\psi(t_1,t_2)=\dfrac{2^{\nu}\sqrt{\pi}t_2^{\nu}}{|b|\Gamma\left(\frac{1}{2}-\nu\right)}(b^2t_1^2-t_2^2)^{-\nu-\frac{1}{2}}\eta(b^2t_1^2-t_2^2)$.

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