Mathematical Modelling and Analysis 2005. Pages 193–197
Proceedings of the 10^{th} International Conference MMA2005&CMAM2, Trakai
 ISBN 9986-05-924-0

NONLOCAL PROBLEM FOR THE SYSTEM OF NONLINEAR DIFFERENTIAL EQUATIONS WITH SEPARATED BOUNDARY CONDITIONS

R. ČIUPAILA[1] and M. SAPAGOVAS[2]

[1] *Vilnius Gediminas Technical University*
Saulėtekio al. 11, LT-10223 Vilnius, Lithuania
[2] *Institute of Mathematics and Informatics*
Akademijos 4, LT-08663 Vilnius, Lithuania
E-mail: `reciup@lrs.lt; mifosap@ktl.mii.lt`

Abstract. The system of two nonlinear first order differential equations with separated boundary conditions and nonlocal integral condition is considered. The system simulates a shape of free surface of micro volume liquid metal drop with non fixed radius on the horizontal plane. The boundary problem is brought to the initial one. The iterative process accounting the curve of the crest of the drop, depending on some geometrical and mechanical parameters, and the radius of adhesion of the drop on the plane, is proposed.
Key words: system of differential equations, separated boundary conditions, nonlocal condition

1. Introduction

The problem investigated in this paper provides the numerical modeling of the free surface of the liquid metal drop on the horizontal plane. The wide range of simulations of liquid drops' shape behavior was proposed in [6].

In applications several parametrical and nonparametrical models were used for simulation of micro volume liquid metal drop shaped contacts, for example, in the telecommunication installments, providing the calculation of the curve of the drop crest. The investigation of liquid drop equations is still urgent from mathematical point of view as well widely used in technical, physical and chemical applications [4, 5, 7, 9]. As a specific feature of the model of the drop on the plane investigated earlier [2, 8], the specific treatment of the plane was used providing radius a of the adhesion of the base of drop to the plane was given (fixed radius).

In the paper [3] the nonlocal problem of the system of three ordinary differential equations of the first order modelling the drop was investigated:

$$\begin{cases} \dfrac{du}{ds} = \sin\varphi, \\ \dfrac{dr}{ds} = \cos\varphi, \\ \dfrac{d\varphi}{ds} = Ku - \dfrac{1}{r}\sin\varphi - \lambda, \end{cases} \tag{1.1}$$

where $s \in [0, 1]$. Boundary conditions

$$u(1) = 0, \quad r(0) = 0, \quad \varphi(0) = 0, \tag{1.2}$$

and integral nonlocal condition of volume of the drop

$$2\pi \int_0^1 ur\cos\varphi \, ds = V_0 \tag{1.3}$$

were added. Similar axisymmetric model of the drop was used in [7]. The similar system for nonconstraint problem used in [5].

Here, respectively, u is height and r is radius of every point on the curve of the crest of drop, φ is an angle of the tangent and axis Ox at that point. λ is unknown constant (Lagrange multiplier), V_0 – volume of the drop, K – parameter including specific tension of the surface of liquid, material density and gravity.

2. Problem Formulation

In this paper the system of two nonlinear equations modeling the shape of the crest of the drop is investigated. This system is obtained remaking the system of three differential equations (1.1) – (1.3).

The main purpose of this paper is to find an effective method for the solution of the problem with non fixed radius of the drop. In this case the liquid on the plane spreads according to the physical laws of minimum energy and optimal surface, depending on the wetting angle φ_1 which is a constant individual for the respective materials of liquid in the drop and the plane. Thus φ_1 becomes a parameter of the problem.

The differential model leads to the nonlocal problem for the system of two nonlinear differential equations with separated boundary conditions.

Providing

$$\frac{du}{ds} = \frac{du}{d\varphi}\frac{d\varphi}{ds}, \quad \frac{dr}{ds} = \frac{dr}{d\varphi}\frac{d\varphi}{ds}$$

and putting these expressions to the equations (1.1) and integral condition (1.3), we get the system of two parametrical equations

$$\begin{cases} \dfrac{du}{d\varphi} = \dfrac{\sin\varphi}{Ku - \frac{1}{r}\sin\varphi - \lambda}, \\ \dfrac{dr}{d\varphi} = \dfrac{\cos\varphi}{Ku - \frac{1}{r}\sin\varphi - \lambda} \end{cases} \tag{2.1}$$

with the separated boundary conditions

$$u(\varphi_1) = 0, \quad r(0) = 0, \tag{2.2}$$

taken in the different end-points of the interval $\varphi \in [0, \varphi_1]$, and nonlocal condition

$$2\pi \int_0^a ur\, dr = V_0\,. \tag{2.3}$$

The solution of the problem then is given by $(u(\varphi), r(\varphi), \lambda, a)$, where $a = r(\varphi_1)$.

To solve the problem (2.1) – (2.3) we bring this boundary value problem to the equivalent initial value problem (or the *Cauchy* problem) (2.1), (2.3) with the initial conditions

$$u(\varphi_1) = 0, \quad r(\varphi_1) = a, \tag{2.4}$$

as it is used, for example, in [1]. Radius a of the base of drop is non fixed, then the Cauchy problem (2.1), (2.3), (2.4) depends on the unknown parameter $a = r(\varphi_1)$, i.e. the solution is $u = u(\varphi, a), r = r(\varphi, a)$. The parameter a should satisfy the condition

$$r(0, a) = 0.$$

In the paper [3] it was shown that the algebraic connection for λ to the other parameters of the problem can be obtained for the problem with fixed radius:

$$\frac{\lambda}{2}a^2 + a\cos\varphi_1 - \frac{K}{V_0} = 0. \tag{2.5}$$

Getting λ from (2.5) we put it to (2.1). From (2.5) it follows that

$$a = -\frac{1}{\lambda}\cos\varphi_1 \pm \sqrt{\frac{1}{\lambda^2}\cos^2\varphi_1 + \frac{K}{\lambda\pi}V_0} \tag{2.6}$$

for sharp as well as obtuse wetting angle φ_1 $(\cos\varphi_1 > 0)$. The uniqueness of the positive radius a was also proven in [3].

Condition (2.6) binds two parameters of the problem a and φ_1, one of which is unknown. In [3] a was fixed parameter while φ_1 was obtained as a result of calculation.

For the problem (2.1), (2.3), (2.4) parameter φ_1 is given while a is unknown. Selfdependense of parameters a and φ_1 allows us to use condition (2.6) for the problem (2.1), (2.3), (2.4).

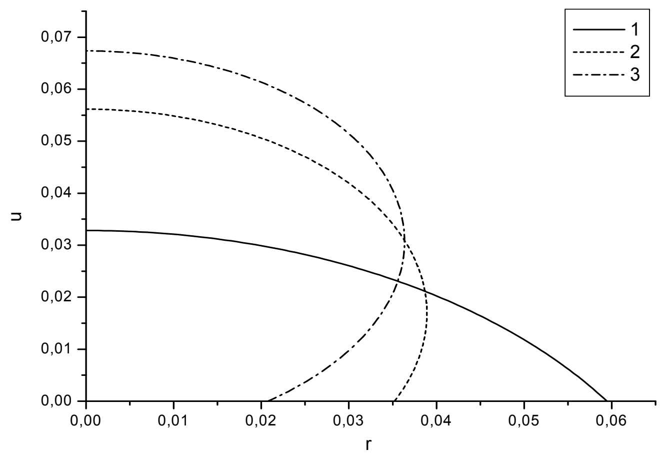

Figure 1. Shape of the drop according wetting angle φ_1.

3. Numerical Experiment

The two stage method was proposed for the numerical solution of the problem.

For the values of radius a the interval $[a_1, a_2]$ was taken, providing condition $r(0, a_1) < 0$ but $r(0, a_2) > 0$. Iteration process was based on shooting and interval bisection methods. The aim was to get value of radius a, which approximetely satisfies condition $r(0, a) = 0$.

For each division value when using interval bisection method, the Runge – Kutta method of 4^{th} order was used to solve the problem (2.1), (2.3), (2.4). Such approach was used, for example, in [7]. As stressed there, the numerical solution may be then regarded as beeing very accurate, except for the region close to the axis ($r = 0$).

We shall give a note concerning some specific features of the algorithm. For right-hand sides of equations of the system (2.1)

$$f_1 = \frac{\sin\varphi}{Ku - \frac{1}{r}\sin\varphi - \lambda}, \quad f_2 = \frac{\cos\varphi}{Ku - \frac{1}{r}\sin\varphi - \lambda}$$

the derivatives

$$\left|\frac{\partial f_i}{\partial u}\right|, \left|\frac{\partial f_i}{\partial r}\right|, \quad i = 1, 2$$

are bounded in each of the intervals $[\varepsilon, \varphi_1]$, there ε is desirable small number.

The derivatives as well f_1 and f_2 contain the ratio

$$\frac{\sin\varphi}{r}, \quad \frac{\sin\varphi}{r^2}, \tag{3.1}$$

with the property of possible unboundness when $\varphi \to 0\,(r \to 0)$.

Therefore it was important to calculate with hight accuracy the values of ratio in (3.1) when $\varphi \to 0$. In all cases of numerical experiment, the values of ratio in (3.1) turned to zero, when $\varphi \to 0$ $(r(0,a) \to 0)$. Any way, to avoid the possible loss of accuracy we applied the Runge-Kutta procedure from the right side of the interval $(0, \varphi_1]$. Similar approch when integration process was starting from the maximum value of radius r towards the top of the drop is used in [7].

The similar effect we get when investigating the expression of denominator $Ku - \dfrac{\sin\varphi}{r} - \lambda$ of $f_1,\ f_2$, which may turn to zero. Calculations show that the ratio $\dfrac{\sin\varphi}{r} \to 0$ when $r \to 0$. In addition in [3] it was shown that because $Ku - \dfrac{\sin\varphi}{r} - \lambda \le 0$ along the boundary curve, in surroundings of the solution u, r, λ, the denominator never turns to zero. The effectiveness of the algorithm is illustrated by the series of numerical results. The shape of the sessile drop on the plane with non fixed radius of adhesion is obtained, respectively according to the wetting angle φ_1 (see Fig. 1), the volume of the drop V_0, and to overweight K. In the Fig.1 lines 1,2,3 indicate the curves of drop crest for the wetting angle equal 1[rad], 2[rad] and 2.5[rad] respectively, $V_0 = 0.0002, K = 28.8$.

References

[1] M. Bachvalov, N. Zhidkov and G. Kobelkov. *Chislennye metody.* Nauka, Moscow, 1987.

[2] R. Čiegis and R. Čiupaila. On the variational-difference method for one problem of conditional minimization. *Liet. Mat. Rink.*, **30**(4), 810 – 822, 1990.

[3] R. Čiupaila and M. Sapagovas. Solution of the system of parametric equations of the sessile drop. *Mathematical Modelling and Analysis*, **7**(2), 201 – 206, 2002.

[4] B. Concus, R. Finn and M. Weislogel. Capillarly surfaces in an exotic container: results from space experiments. *J. Fluid Mech.*, **394**, 119 – 135, 1999.

[5] A. Elcrat and R. Treinen. Numerical results for floating drops. discrete and continues dynamical systems, 241 – 249, 2005. http://AIMsciences.org

[6] R. Finn. *Equilibrium Capillary Surfaces.* Springer-Verlag, N.Y., 1986.

[7] N. Mangiavacchi, A. Castelo, M.F. Tome, J.A. Cuminato, M.L. Bambozi de Oliveira and A. McKee. An effective implementation of surface tension using the marker and cell method for axisymmetric and planar flows. *SIAM J. Sci. Comput.*, **26**(4), 1340 – 1356, 2005.

[8] M. Sapagovas. On the investigation of the convergence of finite difference method for the Neumann boundary problem of the surface of the drop. *Differ. Equations*, **37**(7), 1019 – 1025, 2001.

[9] B. Shapiro, H. Moon, R.L. Garell and C.J. Kim. Equilibrium behavior of sessile drops under surface tension applied external fields, and material variations. *Journal of Applied Physics*, **93**(9), 5794 – 5810, 2003.