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APPROXIMATE SOURCE CONDITIONS IN REGULARIZATION AND AN APPLICATION TO MULTIPLICATION OPERATORS

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Abstract. We present a new approach for finding convergence rates in Tikhonov regularization based on the consideration of approximate source conditions and corresponding distance functions. In this context, we distinguish logarithmic, power and exponential decay rates for the distance functions and their consequences. An application to multiplication operators is given. Moreover, some ideas of generalization are mentioned concerning the fact that the benchmark of the distance functions can be shifted.

Key words: linear ill-posed problems, Tikhonov regularization, approximate source conditions, distance functions, convergence rates, multiplication operators

1. Introduction

Let X and Y be infinite dimensional Hilbert spaces, where the symbol $\|\cdot\|$ denotes the generic norm in both spaces as well as associated operator norms. In this paper, we are going to study ill-posed linear operator equations

$$Ax = y, \quad x \in X,\ y \in Y \tag{1.1}$$

with injective and bounded linear operators $A : X \to Y$ having a non-closed range $\mathcal{R}(A)$, for which the stable approximate solution requires regularization methods. In the sequel we focus on the Tikhonov method (see, e.g., [1, 2, 4, 10]) as a standard approach. Let $x_0 \in X$ be the unique solution of equation (1.1) for an exact right-hand side $y = Ax_0 \in Y$. Instead of y we assume to know the noisy data element $y^\delta \in Y$ with noise level $\delta > 0$ and

$$\|y^\delta - y\| \le \delta.$$

We distinguish regularized solutions

$$x_\alpha \;=\; (A^*A+\alpha I)^{-1} A^*\, y$$

with regularization parameter $\alpha > 0$ in the case of noise-free data and

$$x_\alpha^\delta \;=\; (A^*A+\alpha I)^{-1} A^* y^\delta$$

in the case of noisy data.

Here we call the noise-free error function

$$f(\alpha) \;:=\; \|x_\alpha - x_0\| \;=\; \|\alpha\,(A^*A+\alpha I)^{-1}\, x_0\| \qquad (\alpha > 0) \tag{1.2}$$

profile function for fixed A and x_0. Taking into account the noise level δ this function determines the total regularization error of Tikhonov regularization

$$e(\alpha,\delta) \;:=\; \|x_\alpha^\delta - x_0\| \le \|x_\alpha - x_0\| + \|x_\alpha^\delta - x_\alpha\| \tag{1.3}$$

with the well-known estimate

$$e(\alpha,\delta) \;\le\; f(\alpha) + \|\,(A^*A+\alpha I)^{-1} A^*(y^\delta - y)\| \;\le\; f(\alpha) + \frac{\delta}{2\sqrt{\alpha}}. \tag{1.4}$$

2. Convergence Rates for Tikhonov Regularization Based on Approximate Source Conditions

To obtain convergence rates for the Tikhonov regularization and other linear regularization methods, in the recent years general source conditions

$$x_0 \;= \varphi(A^*A)\, w \qquad (w \in X) \tag{2.1}$$

with index functions φ in the sense of [8] were used (see also [7, 9]). For the Tikhonov regularization method the following proposition can be derived from the literature.

Proposition 1. *We assume that (2.1) holds and the index function $\varphi(t)$ is concave for $0 \le t \le \hat{t}$ with some positive constant $0 < \hat{t} \le \|A\|^2$. Then the profile function (1.2) satisfies an estimate*

$$f(\alpha) \;=\; \|\alpha\,(A^*A+\alpha I)^{-1}\, \varphi(A^*A)\, w\| \;\le\; K\, \varphi(\alpha)\, \|w\| \qquad (0 < \alpha \le \overline{\alpha}) \tag{2.2}$$

for some $\overline{\alpha} > 0$ and a constant $K \ge 1$ which is one for $\hat{t} = \|A\|^2$. Hence, we have

$$e(\alpha,\delta) \;\le\; K\, \varphi(\alpha)\, \|w\| + \frac{\delta}{2\sqrt{\alpha}} \qquad (0 < \alpha \le \overline{\alpha}) \tag{2.3}$$

for the total regularization error.

In this paper, we present an alternative approach for finding estimates of the form (2.2) and (2.3) and consequently convergence rates for the Tikhonov regularization on the basis of the following lemma. This approach avoids the use of explicit general source conditions (2.1). For a couple of more details and applications concerning that method we refer to the papers [3, 5].

Lemma 1. *Based on the distance function*

$$d(R) := \inf\{\|x_0 - A^* v\| \,:\, v \in Y,\ \|v\| \le R\} \tag{2.4}$$

that measures for x_0 the violation of the specific source condition

$$x_0 = A^* v_0 \qquad (v_0 \in Y,\ \|v_0\| \le R)\,, \tag{2.5}$$

we obtain

$$f(\alpha) = \|x_\alpha - x_0\| \le d(R) + \frac{\sqrt{\alpha}}{2} R \tag{2.6}$$

for all $\alpha > 0$ and $R \ge 0$ as an estimate for the profile function of regularized solutions in Tikhonov regularization.

Proof. Let $v \in Y$ with $\|v\| \le R$. Then based on formula (1.2) we can estimate by the triangle inequality as follows:

$$\begin{aligned} f(\alpha) = {} & \|\alpha\,(A^*A+\alpha I)^{-1}\,x_0 - \alpha\,(A^*A+\alpha I)^{-1}\,A^*v + \alpha\,(A^*A+\alpha I)^{-1}\,A^*v\| \\ \le {} & \|\alpha\,(A^*A+\alpha I)^{-1}\,(x_0 - A^*v)\| + \|\alpha\,(A^*A+\alpha I)^{-1}\,A^*v\| \\ \le {} & \alpha\,\|\,(A^*A+\alpha I)^{-1}\,\|\,\|x_0 - A^*v\| + \alpha\,\|\,(A^*A+\alpha I)^{-1}\,A^*\|\,\|v\| \\ \le {} & \alpha\,\tfrac{1}{\alpha}\,\|x_0 - A^*v\| + \alpha\,\tfrac{1}{2\sqrt{\alpha}}\,\|v\| \le \|x_0 - A^*v\| + \tfrac{1}{2}\sqrt{\alpha}\,R\,. \end{aligned}$$

Since the inequality $f(\alpha) \le \|x_0 - A^*v\| + \frac{1}{2}\sqrt{\alpha}\,R$ thus obtained remains valid if we substitute $\|x_0 - A^*v\|$ by $\inf\{\|x_0 - A^* v\| : v \in Y,\ \|v\| \le R\}$, we immediately find the inequality (2.6). This proves the lemma. ■

Evidently, for every $x_0 \in X$ the nonnegative distance function $d(R)$ depending on the radius $R \in [0,\infty)$ is well-defined and non-increasing with $\lim_{R\to\infty} d(R) = 0$ as a consequence of the injectivity of A and $\overline{\mathcal{R}(A^*)} = X$. The distance function $d(R)$ expresses the behavior of x_0 with respect to the benchmark condition (2.5).

Note that an estimate

$$f(\alpha) \le \sqrt{d^2(R) + \alpha\,R^2} \le d(R) + \sqrt{\alpha}\,R \tag{2.7}$$

similar to (2.6) directly follows from Theorem 6.8 in [1]. The proof of this theorem, however, is completely different from that of our Lemma 1. The inequalities (2.7) were basic for convergence rate results presented in the papers [5] and [6], but it is evident that the same results can also be derived from Lemma 1.

Example 1 [Logarithmic type decay]. If $d(R)$ decreases to zero very slowly as $R \to \infty$, the resulting rate for $f(\alpha) \to 0$ as $\alpha \to 0$ is also very slow. Here, we consider the family of distance functions

$$d(R) \le K\,(\ln R)^{-p}, \quad \underline{R} \le R < \infty \tag{2.8}$$

for some constants $\underline{R} > 0$, $K > 0$ and for parameters $p > 0$. By setting $R := \alpha^{-\kappa}$ $(0 < \kappa < \frac{1}{2})$ and taking into account that $\alpha = o\left((\ln(1/\alpha))^{-p}\right)$ as $\alpha \to 0$ we have from Lemma 1 and (2.8)

$$f(\alpha) \le \widetilde{K}\, (\ln(1/\alpha))^{-p} \qquad (0 < \alpha \le \overline{\alpha})$$

for some $\overline{\alpha} > 0$ and a constant $\widetilde{K} > 0$. Then by using the a priori parameter choice $\alpha(\delta) = c_0\, \delta^{\chi}$ with some exponent $0 < \chi < 2$ we obtain the logarithmic convergence rate

$$e(\alpha(\delta), \delta) = \mathcal{O}\left((\ln(1/\delta))^{-p}\right) \quad \text{as} \quad \delta \to 0$$

discussed, e.g., in [7] with respect to general source conditions (2.1) and corresponding logarithmic index functions φ.

Example 2 [Power type decay]. If $d(R)$ behaves as a power of R, i.e.,

$$d(R) \le K\, R^{\frac{\gamma}{\gamma-1}} \qquad (\underline{R} \le R < \infty) \tag{2.9}$$

with parameters $0 < \gamma < 1$ and constants $\underline{R} > 0$, $K > 0$, then by setting $R := \alpha^{\frac{\gamma-1}{2}}$ we derive from Lemma 1 an estimate

$$f(\alpha) \le \widetilde{K}\, \alpha^{\frac{\gamma}{2}} \qquad (0 < \alpha \le \overline{\alpha})$$

for some $\overline{\alpha} > 0$ and a constant $\widetilde{K} > 0$. Here, the negative exponent $\gamma/(1-\gamma)$ in (2.9) attains all positive values when γ covers the open interval $(0,1)$. If the a priori parameter choice $\alpha(\delta) \sim \delta^{\frac{2}{1+\gamma}}$ is used, we find from (1.4)

$$e(\alpha(\delta), \delta) = \mathcal{O}\left(\delta^{\frac{\gamma}{1+\gamma}}\right) \quad \text{as} \quad \delta \to 0\,. \tag{2.10}$$

For $0 < \gamma < 1$ formula (2.10) includes all Hölder convergence rates that are slower than the rate $\mathcal{O}(\sqrt{\delta})$ which characterizes the source condition (2.5).

Example 3 [Exponential type decay]. Even if $d(R)$ falls exponentially, i.e.,

$$d(R) \le K\, \exp(-c\, R^q) \qquad (\underline{R} \le R < \infty)$$

for parameters $q > 0$ and constants $\underline{R} > 0$, $K > 0$ and $c \ge \frac{1}{2}$, the convergence rate $\mathcal{O}(\sqrt{\delta})$ cannot be obtained on the basis of Lemma 1. From (2.6) we have with $R := (\ln(1/\alpha))^{1/q}$ the estimate

$$f(\alpha) \le \widetilde{K}\, (\ln(1/\alpha))^{1/q}\, \sqrt{\alpha} \qquad (0 < \alpha \le \overline{\alpha})$$

for some $\overline{\alpha} > 0$ and a constant $\widetilde{K} > 0$. Hence with $\alpha(\delta) \sim \delta$ we derive a convergence rate

$$e(\alpha(\delta), \delta) = \mathcal{O}\left((\ln(1/\delta))^{1/q}\, \sqrt{\delta}\right) \quad \text{as} \quad \delta \to 0,$$

which is only a little slower than $\mathcal{O}(\sqrt{\delta})$.

In the paper [6] one can find sufficient conditions for the Examples 1 and 2 formulated as range inclusions with respect to $\mathcal{R}(A^*)$ and also examples of compact operators A that satisfy such conditions. On the other hand, the cases of Examples 2 and 3 are illustrated below in the context of non-compact multiplication operators.

3. An Application to Multiplication Operators

In this section we consider $X = Y = L^2(0,1)$ and specify A as a multiplication operator

$$[A\,x]\,(t) \;=\; m(t)\,x(t) \qquad (0 \le t \le 1)$$

defined by a multiplier function $m \in L^\infty(0,1)$ with essential zeros such that $\mathcal{R}(A)$ is not closed. For simplicity, let us assume

$$x_0(t) \;=\; 1 \qquad (0 \le t \le 1) \tag{3.1}$$

in the following two situations. Then we can formulate the following results. For proofs we refer to [5].

Proposition 2. *For the solution (3.1) of equation (1.1) and the multiplier function*

$$m(t) \;=\; t \qquad (0 \le t \le 1)$$

we have with some constant $\underline{R} > 0$ *an estimate of the form*

$$d(R) \;\le\; \frac{\sqrt{2}}{R} \qquad (\underline{R} \le R < \infty)$$

for the distance function (2.4) of the pure multiplication operator A.

The situation of Proposition 2 corresponds with the case $\gamma = \frac{1}{2}$ in Example 2 and yields $f(\alpha) = \mathcal{O}(\sqrt[4]{\alpha})$ implying the Hölder rate $\mathcal{O}(\sqrt[3]{\delta})$, which is order optimal in that situation.

Proposition 3. *For the solution (3.1) of equation (1.1) and the multiplier function*

$$m(t) \;=\; \sqrt{t} \qquad (0 \le t \le 1)$$

we have with some constant $\underline{R} > 0$ *an estimate of the form*

$$d(R) \;\le\; \exp\left(-\frac{1}{2}R^2\right) \qquad (\underline{R} \le R < \infty)$$

for the distance function (2.4) of the pure multiplication operator A.

Obviously, the situation of Proposition 3 corresponds with the case $c = \frac{1}{2}$ and $q = 2$ in Example 3 and yields $f(\alpha) = \mathcal{O}\left(\sqrt{\left(\ln\frac{1}{\alpha}\right)\,\alpha}\right)$. For that situation our alternative approach does not provide us with the order optimal convergence rate $f(\alpha) = \mathcal{O}\left(\sqrt{\alpha}\right)$. This is a drawback of the suggested method based on Lemma 1. By construction of this approach we cannot obtain higher order rates $f(\alpha) = \mathcal{O}\left(\alpha^\mu\right)$ with $\mu \ge \frac{1}{2}$. This is some kind of limitation for the presented technique.

4. Shifting the Benchmark

In order to overcome the limitation mentioned above, we can replace the specific source condition (2.5) by a general source condition (2.1) with some index function φ. Consequently we have to consider the appropriate distance function

$$\widetilde{d}(R) := \inf \{ \|x_0 - \varphi(A^*A)\, w\| \,:\, w \in X,\ \|w\| \le R \}$$

with a shifted benchmark. Here, $\widetilde{d}(R)$ measures the violation of x_0 with respect to the shifted benchmark. Based on formula (2.2) an analogue of Lemma 1 holds at least for any concave index function φ. Then for the case $\varphi(t) = t$, which corresponds with the saturation of Tikhonov regularization, instead of (2.6) we have the estimate

$$f(\alpha) \le \widetilde{d}(R) + \alpha R$$

for all $\alpha > 0$ and $R \ge 0$. This may yield all convergence rates which are slower than $f(\alpha) = \mathcal{O}(\alpha)$ provided that a sufficiently rapid decay of $\widetilde{d}(R) \to 0$ as $R \to \infty$ occurs. However, it is forthcoming work of the author to study the decay behavior of $\widetilde{d}(R)$ in detail for examples.

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