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NUMERICAL METHOD FOR SINGULARLY PERTURBED ELLIPTIC PROBLEM WITH THE CONCENTRATED SOURCE IN A STRIP

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Abstract. A boundary value problem for an elliptic equation with a small parameter before the highest derivatives in a strip is considered. A method of reduction of problem to a system of ordinary differential equations on a finite interval is investigated. To numerical solution of problem the piecewise uniform mesh condensing along the strip and in a neighbourhood of the concentrated source is used.

Key words: singularly perturbed problems, transfer of the boundary condition from infinity, difference scheme, asymptotic series

1. The Method of Lines for an Elliptic Problem

Consider a boundary value problem

$$\varepsilon^2\frac{\partial^2 u(x,y)}{\partial x^2}+\varepsilon^2\frac{\partial^2 u(x,y)}{\partial y^2}-c(x,y)u(x,y)=f(x,y), \tag{1.1}$$

$$\varepsilon\frac{\partial u(+0,y)}{\partial x}-\varepsilon\frac{\partial u(-0,y)}{\partial x}=-Q(y),$$

$$u(x,0)=\phi_1(x),\quad u(x,1)=\phi_2(x),\quad \lim_{x\to\pm\infty}u(x,y)=0$$

in a strip $D=\{-\infty<x<\infty,\ 0\le y\le 1\}$.

Suppose, that functions c and f are sufficiently smooth on D,

$$\varepsilon\in(0;1],\ \ c(x,y)\ge\alpha>0,\ \ \lim_{x\to\pm\infty}f(x,y)=0,$$

$$\lim_{x\to-\infty}c(x,y)=c_1(y),\ \lim_{x\to\infty}c(x,y)=c_2(y),\ \lim_{x\to\pm\infty}\phi_i(x)=0,\ i=1,2.$$

We use the following norms for functions, vectors and matrices:

$$\|g(x)\| = \max_x |g(x)|, \quad \|g(x,y)\| = \max_{x,y} |g(x,y)|,$$

$$\|\boldsymbol{x}\| = \max_i |x_i|, \quad \|\mathbf{A}\| = \max_i \sum_{j=1}^{n} |\mathbf{A}_{ij}|, \quad 1 \le i \le n.$$

Solution of problem (1.1) has boundary layers along the strip on y and in a neighbourhood of the concentrated source on x.

Lemma 1. *Let $u(x,y)$ is the solution of problem* (1.1). *Then*

$$\|u(x,y)\| \le \frac{\|Q(y)\|}{2\sqrt{\alpha}} exp(-\sqrt{\alpha}\varepsilon^{-1}|x|) + \frac{\|f(x,y)\|}{\alpha} + \max_i \|\phi_i(x)\|.$$

Using method of lines we reduce an elliptic problem to the system of ordinary differential equations on an infinite interval. To take into account boundary layers on value y, we use nonuniform mesh [5]

$$\Omega_y = \{y_j : 0 \le j \le M\}, \quad \sigma = \min\{1/4, \varepsilon \ln M\},$$

$$y_j = \begin{cases} 4j\sigma/M, & 0 \le j \le M/4, \\ \sigma + 2\,(j - M/4)\,(1-2\sigma)/M, & M/4 \le j \le 3M/4, \\ 1 - \sigma + 4\,(j - 3M/4)\,\sigma/M, & 3M/4 \le j \le M \end{cases}$$

and approximate the derivative on this mesh. Let the difference scheme has the form:

$$\varepsilon^2 \frac{d^2 V_j}{dx^2} + \varepsilon^2 \Lambda_{yy,j} V - c(x,y_j) V_j = f(x,y_j), \quad 0 < j < M,$$

$$\varepsilon V_j'(+0) - \varepsilon V_j'(-0) = -Q(y_j),$$

$$V_0(x) = \phi_1(x), \quad V_M(x) = \phi_2(x), \quad \lim_{x\to\pm\infty} V_j = 0,$$

$$\Lambda_{yy,j} V = \frac{h_j(V_{j+1} - V_j) - h_{j+1}(V_j - V_{j-1})}{h_j h_{j+1}(h_j + h_{j+1})/2},$$

then, according to [5], the following estimate is valid:

$$\max_j \max_x |U_j(x) - V_j(x)| \le \frac{C}{M^2} \ln^2 M, \quad U = [u]_{\Omega_y}.$$

So, we get a system of differential equations on an infinite interval:

$$\varepsilon^2 \boldsymbol{V}''(x) - \mathbf{A}(x)\boldsymbol{V}(x) = \boldsymbol{F}(x), \tag{1.2}$$

$$\varepsilon \boldsymbol{V}'(+0) - \varepsilon \boldsymbol{V}'(-0) = -\boldsymbol{Q}, \quad \lim_{x\to\pm\infty} \boldsymbol{V}(x) = \mathbf{0},$$

where matrix $\mathbf{A}(x)$ is a three-diagonal $n\times n$ nonsingular M-matrix, $n = M-1$,

$$\sum_{j=1}^{n} \mathbf{A}_{ij}(x) \geq \alpha > 0, \quad \mathbf{A}_{ij}(x) \leq 0,\ j \neq i,\ i = 1,\dots,n,$$

$$\lim_{x\to-\infty} \mathbf{A}(x) = \mathbf{A}_1, \quad \lim_{x\to\infty} \mathbf{A}(x) = \mathbf{A}_2, \quad \lim_{x\to\pm\infty} \boldsymbol{F}(x) = \mathbf{0}.$$

If $\alpha > \dfrac{8(1-4\sigma)(1-3\sigma)}{M^2}$, then $\sum_{i=1}^{n} \mathbf{A}_{ij}(x) > 0$. The existence and uniqueness of solution of problem (1.2) are proved.

Lemma 2. *Let* $\boldsymbol{V}(x)$ *is the solution of problem* (1.2)*. Then*

$$\|\boldsymbol{V}(x)\| \leq \frac{1}{\alpha} \max_x \|\boldsymbol{F}(x)\| + \frac{\|\mathbf{Q}\|}{2\sqrt{\alpha}} \exp\left(-\sqrt{\alpha}\varepsilon^{-1}|x|\right).$$

Lemma 3. *Let* $\boldsymbol{V}(x)$ *be the solution of problem* (1.2)*. Then there is a constant* C*, such that*

$$|V_i^{(k)}(x)| \leq C\left[1 + \frac{1}{\varepsilon^k} \exp(-\sqrt{\alpha}\varepsilon^{-1}|x|)\right], \quad i = 1,\dots,n,\ k \geq 1.$$

2. Reduction of the Problem to Finite Interval

To solve problem (1.2) numerically it is necessary to reduce it to a problem specified on a finite interval. For this purpose we extract sets of solutions satisfying the limiting conditions at infinity. These sets are defined as systems of first order differential equations [1]:

$$\varepsilon\boldsymbol{V}'(x) + \mathbf{B}_1(x)\boldsymbol{V}(x) = \boldsymbol{\beta}_1(x), \quad x < 0, \tag{2.1}$$

$$\varepsilon\boldsymbol{V}'(x) + \mathbf{B}_2(x)\boldsymbol{V}(x) = \boldsymbol{\beta}_2(x), \quad x > 0, \tag{2.2}$$

where matrices $\mathbf{B}_1(x)$, $\mathbf{B}_2(x)$ are solutions of singular problems for the matrix Riccati equations:

$$\varepsilon\mathbf{B}_1'(x) - \mathbf{B}_1^2(x) + \mathbf{A}(x) = \mathbf{0},\ \lim_{x\to-\infty} \mathbf{B}_1(x) = -\sqrt{\mathbf{A}_1}, \tag{2.3}$$

$$\varepsilon\mathbf{B}_2'(x) - \mathbf{B}_2^2(x) + \mathbf{A}(x) = \mathbf{0},\ \lim_{x\to\infty} \mathbf{B}_2(x) = \sqrt{\mathbf{A}_2}, \tag{2.4}$$

and the vector-functions $\boldsymbol{\beta}_1(x)$, $\boldsymbol{\beta}_2(x)$ are solutions of singular Cauchy problems:

$$\varepsilon\boldsymbol{\beta}_1'(x) - \mathbf{B}_1(x)\boldsymbol{\beta}_1(x) = \boldsymbol{F}(x),\ \lim_{x\to-\infty} \boldsymbol{\beta}_1(x) = \mathbf{0}, \tag{2.5}$$

$$\varepsilon\boldsymbol{\beta}_2'(x) - \mathbf{B}_2(x)\boldsymbol{\beta}_2(x) = \boldsymbol{F}(x),\ \lim_{x\to\infty} \boldsymbol{\beta}_2(x) = \mathbf{0}. \tag{2.6}$$

Using the extracted sets we reduce problem (1.2) to the problem on a finite interval:

$$\begin{aligned}
&\varepsilon^2\boldsymbol{V}''(x)-\mathbf{A}(x)\boldsymbol{V}(x)=\boldsymbol{F}(x),\ L_1<x<L_2,\ x\neq 0, \\
&\varepsilon\boldsymbol{V}'(+0)-\varepsilon\boldsymbol{V}'(-0)=-\boldsymbol{Q}, \\
&\varepsilon\boldsymbol{V}'(L_1)+\mathbf{B}_1(L_1)\boldsymbol{V}(L_1)=\boldsymbol{\beta}_1(L_1),\ L_1<0, \\
&\varepsilon\boldsymbol{V}'(L_2)+\mathbf{B}_2(L_2)\boldsymbol{V}(L_2)=\boldsymbol{\beta}_2(L_2),\ L_2>0.
\end{aligned} \tag{2.7}$$

It is easy to prove that solutions of problems (1.2) and (2.7) coincide for all $x\in[L_1;L_2]$, where $L_1<0,\ L_2>0$ can take any finite arbitrary values.

The matrices $\mathbf{B}_1(x)$, $\mathbf{B}_2(x)$ and the vector-functions $\boldsymbol{\beta}_1(x)$, $\boldsymbol{\beta}_2(x)$ from singular Cauchy problems (2.3)-(2.6) can be found as asymptotic series on parameter ε :

$$\mathbf{B}_k^m(x)=\sum_{i=0}^{m}\mathbf{B}_{k,i}(x)\,\varepsilon^i,\quad \boldsymbol{\beta}_k^m(x)=\sum_{i=0}^{m}\boldsymbol{\beta}_{k,i}(x)\,\varepsilon^i,\quad k=1,2.$$

Then we get recurrent formulas for coefficients $\mathbf{B}_{k,i}$ and $\boldsymbol{\beta}_{k,i}$:

$$\mathbf{B}_{k,0}(x)\mathbf{B}_{k,i}(x)+\mathbf{B}_{k,i}(x)\mathbf{B}_{k,0}(x)=\mathbf{B}'_{k,i-1}(x)-\sum_{j=1}^{i-1}\mathbf{B}_{k,j}(x)\mathbf{B}_{k,i-j}(x), \tag{2.8}$$

$$\mathbf{B}_{1,0}(x)=-\sqrt{\mathbf{A}(x)},\quad \mathbf{B}_{2,0}(x)=\sqrt{\mathbf{A}(x)},$$

$$\mathbf{B}_k(x)\boldsymbol{\beta}_{k,i}(x)=\boldsymbol{\beta}'_{k,i-1}(x),\quad \mathbf{B}_k(x)\boldsymbol{\beta}_{k,0}(x)=-\boldsymbol{F}(x),\quad k=1,2.$$

For formulation of problem (2.7) it is enough to calculate the coefficients $\mathbf{B}_k$ and $\boldsymbol{\beta}_k$ at x equal to L_1 and L_2. To find the square root of M-matrix $\mathbf{A}$ convergent iterative method [2] is used. Since $\mathbf{A}(x)$ is M-matrix, hence [6] it can be written for any finite value L in the form

$$\mathbf{A}(L)=s(\mathbf{I}-\mathbf{P}),\ s>0,\ \mathbf{P}_{ii}>0,\ \mathbf{P}_{ij}\geq 0,\ j\neq i,\ i,j=\overline{1,n},\ \rho(\mathbf{P})<1.$$

Then $\sqrt{\mathbf{A}(L)}=\sqrt{s}(\mathbf{I}-\mathbf{Y}^*)$, where $\mathbf{Y}^*$ denote the limit of the sequence

$$\mathbf{Y}_{k+1}=\frac{1}{2}(\mathbf{P}+\mathbf{Y}_k^2),\ \mathbf{Y}_0=\mathbf{0}. \tag{2.9}$$

Lemma 4. *Let* $\mathbf{A}$ *be a nonsingular M-matrix with diagonal dominance. Then* $\sqrt{\mathbf{A}}$ (2.9) *is a nonsingular M-matrix with a diagonal dominance.*

It follows from Lemma 4 that zero members of asymptotic series for matrices $\mathbf{B}_k$ are M-matrices.

The equation on $\mathbf{B}_k(x)$ (2.8) is continuous Silvester equation which is uniquely solvable for any right side function whenever $\lambda_i(\mathbf{B}_{k,0})+\lambda_j(\mathbf{B}_{k,0})\neq 0$ for any i,j [4]. Since $\mathbf{B}_{k,0}(x)$ are nonsingular M-matrices, hence the real parts of their eigenvalues are positive.

To solve such equations there are orthogonal methods, for example, Bartelse–Stuart's and Golub–Nesh–van–Loan's algorithms [4].

Lemma 5. $\mathbf{B}_k(x)$ *are M-matrices for small* ε,

$$|(\mathbf{B}_k(x))_{ij} - (\mathbf{B}_{k,0}(x))_{ij}| \le C_k\varepsilon, \quad i,j = 1,\dots,n,\ k = 1,2.$$

Lemma 6. *Let* $\tilde{\mathbf{B}}_1(x)$, $\tilde{\mathbf{B}}_2(x)$ *are solutions of problems* (2.3), (2.4) *with the matrix* $\tilde{\mathbf{A}}(x)$ *such that*

$$\|\mathbf{A} - \tilde{\mathbf{A}}\| \le \Delta,\ \|\mathbf{A}_k^{1/2} - \mathbf{A}_k^{\tilde{1/2}}\| \le \delta,$$

$$(-1)^k \sum_{i=1}^{n} (\mathbf{B}_k(x))_{ij} \ge \tilde{\alpha} > 0,\ (-1)^k \sum_{j=1}^{n} (\tilde{\mathbf{B}}_k(x))_{ij} \ge \tilde{\alpha} > 0.$$

Then

$$\|\mathbf{B}_k - \tilde{\mathbf{B}}_k\| \le C(\delta + \Delta),\ k = 1,2.$$

Matrices $\mathbf{B}_k(x)$ and vector-functions $\boldsymbol{\beta}_k(x)$, $k = 1,2$, can be found only approximately. Next we obtain a stability estimate of the solution for problem (2.7) with respect to errors in these coefficients.

Theorem 1. *Let* $\tilde{\boldsymbol{V}}(x)$ *be a solution of problem* (2.7) *with the coefficients* $\tilde{\mathbf{B}}_k(L_k)$, $\tilde{\boldsymbol{\beta}}_k(L_k)$, $k = 1,2$, *such that*

$$(-1)^k \sum_{j=1}^{n} (\tilde{\mathbf{B}}_k(x))_{ij} \ge \tilde{\alpha} > 0,$$

$$\|\mathbf{B}_k(L_k) - \tilde{\mathbf{B}}_k(L_k)\| \le \Delta,\ \ \|\boldsymbol{\beta}_k(L_k) - \tilde{\boldsymbol{\beta}}_k(L_k)\| \le \Delta.$$

Then for all $L_1 \le x \le L_2$, *we get the estimate*

$$\|\boldsymbol{V}(x) - \tilde{\boldsymbol{V}}(x)\| \le C\Delta.$$

The solution of problem (2.7) has a boundary layer in a neighbourhood of the concentrated source. To get a difference scheme with the property of an uniform convergence, we use the following piecewise uniform mesh:

$$\Omega_x = \{x_j : \ \ 0 \le j \le 2N\},\ \ q = \min\{1/2, \varepsilon \ln N\},$$

$$x_j = \begin{cases} L_1 + 2(-L_1 - q)j/N, & 0 \le j \le N/2, \\ -q + 2q\,(j - N/2)\,/N, & N/2 < j \le N, \\ 2q\,(j - N)\,/N, & N < j \le 3N/2, \\ q + 2(L_2 - q)\,(j - 3N/2)\,/N, & 3N/2 < j \le 2N. \end{cases}$$

The difference scheme is given as:

$$2\varepsilon^2 \frac{h_j(\mathbf{V}_{j+1}^h - \boldsymbol{V}_j^h) - h_{j+1}(\boldsymbol{V}_j^h - \boldsymbol{V}_{j-1}^h)}{h_j h_{j+1}(h_j + h_{j+1})} - \mathbf{A}_j \boldsymbol{V}_j^h = \boldsymbol{F}_j,$$

$$\varepsilon \frac{\boldsymbol{V}_{N+1}^h - \boldsymbol{V}_N^h}{h_{N+1}} - \varepsilon \frac{\boldsymbol{V}_N^h - \boldsymbol{V}_{N-1}^h}{h_N} = -\boldsymbol{Q}_j,$$

$$\varepsilon \frac{\boldsymbol{V}_1^h - \boldsymbol{V}_0^h}{h_1} + \mathbf{B}_1(L_1)\boldsymbol{V}_0^h = \boldsymbol{\beta}_1(L_1),$$

$$\varepsilon \frac{\boldsymbol{V}_{2N}^h - \boldsymbol{V}_{2N-1}^h}{h_{2N}} + \mathbf{B}_2(L_2)\boldsymbol{V}_{2N}^h = \boldsymbol{\beta}_2(L_2),$$

where $\|\boldsymbol{V} - \boldsymbol{V}^h\| \le C(N^{-2}\ln^2 N + N^{-1})$ according to [3].

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