Mathematical Modelling and Analysis 2005. Pages 427–431
Proceedings of the 10th International Conference MMA2005&CMAM2, Trakai
 ISBN 9986-05-924-0

ON THE EXISTENCE OF ANALYTIC FUNCTIONS WITH NONVANISHING FINITE DIFFERENCES OF SEVERAL ORDERS IN THE HALF-PLANE

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Abstract. In the present paper multistage finite differences of different orders for functions, analytical in the half-plane, are investigated. The main result is assertion on the absence of functions analytic in the half-plane, which have two nonvanishing finite differences and the difference between their orders exceeds a certain positive number. The proof is based on the properties of univalent functions. In particular, estimations for the derivatives of univalent functions, obtained by I.A. Aleksandrov as the corollary of Louis de Branges theorem about the coefficients of the univalent and normalized in the unit disk functions, are used.
Key words: univalent function, finite difference, half-plane

1. Introduction

Let $F(z)$ be an analytical function in the domain $D(\alpha) = \{|\arg z| < \pi/(2\alpha)\}$, where $\alpha \geq 1$. Let also $\tilde{D}(\alpha) = \{|\arg z| \leq \pi/(2\alpha)\} \setminus \{0\}$. Let us define finite difference of n-th order of function $F(z)$ by recurrent formula

$$\Delta_n [F(z); z, \zeta_1, ..., \zeta_n] = \Delta_1 [\Delta_{n-1} [F(z); z, \zeta_1, ..., \zeta_{n-1}]; z, \zeta_n],$$

$$\Delta_1 [F(z); z, \zeta_1] = F(z+\zeta_1) - F(z), \quad \Delta_0 [F(z); z] \equiv F(z).$$

Note, that if $z \in D(\alpha)$ and $\zeta_1, ..., \zeta_n \in \tilde{D}(\alpha)$, then $z+\zeta_1+...+\zeta_n \in D(\alpha)$. Let $Q_n(D(\alpha))$ be a class of analytical in domain $D(\alpha)$ functions with property $\Delta_n [F(z); z, \zeta_1, ..., \zeta_n] \neq 0$ for any $z \in D(\alpha)$, $\zeta_1, ..., \zeta_n \in \tilde{D}(\alpha)$. Then there arise a question on the existence of functions, which belong simultaneously to all classes $Q_n(D(\alpha))$, $n = 0, 1, 2, \ldots$. In case $\alpha > 1$, the example of such function is given by $F(z) = e^z$. Indeed,

$$\Delta_n [e^z; z, \zeta_1, ..., \zeta_n] = e^z \left(e^{\zeta_1} - 1\right) ... \left(e^{\zeta_n} - 1\right), \;\; z \in D(\alpha), \zeta_1, \ldots, \zeta_n \in \tilde{D}(\alpha).$$

If $\alpha = 0$, then $F(z) = e^z \notin Q_n(D(1))$ for any natural n. The attempts of the authors to find the function, which belongs to all classes $Q_n(D(1))$, did not lead to success and thus a hypothesis about the absence of such function was formulated. We will denote $D(1)$ and $\tilde{D}(1)$ by Π and $\tilde{\Pi}$, respectively.

Further, we will restrict to the case of the half-plane Π. Some properties of functions of the class $Q_n(\Pi)$ were investigated in [5]. In this work we establish the validity of the given hypothesis, i.e., we have proved, that the functions, which belong to all classes $Q_n(\Pi), n = 0, 1, 2, \ldots,$ in fact do not exist. Moreover, we have shown that there is no functions, belonging to two classes $Q_n(\Pi)$, if a difference in their orders exceeds a certain positive number.

Theorem 1. *There is no such function, which simultaneously belongs to two classes $Q_m(\Pi)$ and $Q_k(\Pi)$, where $k - m \geq 5$.*

For the proof of this theorem some definitions and auxiliary assertions will be necessary.

2. Univalent Functions and their Relation with Multistage Finite Differences

Let us remind the determination of the univalent functions, which play important role in the geometric theory of the complex variable functions and realization of the conformal mappings([2, 3, 4]). Analytical in the region D function $F(z)$ is called univalent function in the region D, if $F(z_1) \neq F(z_2)$ for any noncoincident $z_1, z_2 \in D$.

In this work we use corollary of the known Bieberbach theorem about the estimations of coefficients of functions, which are univalent and normed in the unit disk. This corollary was proven by Louis de Branges in 1985 ([1, 2]). Let us denote by $K_1(D)$ a class of univalent in domain D functions. In this paper we will use properties of functions from the class $K_1(D)$ for the solution of problems, connected to the multistage finite differences. In this direction the following lemma plays the key role.

Lemma 1. *The class $K_1(\Pi)$ coincides with class $Q_1(\Pi)$.*

Proof. Let us take the arbitrary two different points ξ, ζ in the half plane Π and let $|\zeta| \geq |\xi|$. Then such a point $\zeta_1 \in \tilde{\Pi}$ exists that $\zeta = \xi + \zeta_1$. Let us assume that the function $F(z)$ belongs to the class $Q_n(\Pi)$. This means that

$$\Delta_1 [F(z); \xi, \zeta_1] = F(\xi + \zeta_1) - F(\xi) = F(\zeta) - F(\xi) \neq 0,$$

and therefore $F(z) \in K_1(\Pi)$. Let us assume now that $F(z) \in K_1(\Pi)$. This means that

$$F(\zeta) - F(\xi) = F(\xi + \zeta_1) - F(\xi) \neq 0.$$

Hence it follows that $F(z) \in Q_1(\Pi)$. Thus, lemma is proved. ■

Lemma 2. *If* $F(z) \in K_1(D)$, *then* $aF(z)+b \in K_1(D)$, *where* a, b *are complex numbers and* $a \neq 0$. *Furthermore, if* $F(z) \in K_1(D)$, *then* $F'(z) \neq 0$ *for any* $z \in D$ [3].

Lemma 3. *If* $F(z) \in Q_n(\Pi)$, *then* $F^{(k)} \in Q_{n-k}(\Pi)$, $\quad k = 1, 2, \ldots, n$ [5].

Lemma 4. *For any fixed natural* n *the following identity with respect to* z *is valid*

$$\sum_{k=0}^{n-1} C_{n-1}^k (k+1+z) = 2^{n-2}(n+1+2z), \tag{2.1}$$

where C_{n-1}^k *are the binomial coefficients.*

Proof. For $n = 1$ the identity (2.1) is correct. Let $n \geq 2$ and

$$\varphi(z) = \sum_{k=0}^{n-1} C_{n-1}^k z^k = (1+z)^{n-1}.$$

Then

$$\varphi'(1) = \sum_{k=0}^{n-1} C_{n-1}^k k = (n-1) 2^{n-2}.$$

Hence

$$\sum_{k=0}^{n-1} C_{n-1}^k (k+1+z) = \sum_{k=0}^{n-1} C_{n-1}^k k + (1+z) \sum_{k=0}^{n-1} C_{n-1}^k$$
$$= (n-1) 2^{n-2} + (1+z) 2^{n-1} = 2^{n-2}(n+1+2z).$$

■

3. Some Auxiliary Estimations

Let us denote by $\tilde{K}_1(\Pi)$ the class of analytical and univalent in half plane Π functions normalized by conditions $F(1) = 0$, $F'(1) = 1$ and let $\tilde{K}_1(E)$ be the class of analytical in the unit disk E (i.e. in the disk $|\omega| < 1$) functions $g(\omega)$, normalized by conditions $g(0) = 0$, $g'(0) = 1$.

Let us note that $\tilde{K}_1(\Pi) \subset K_1(\Pi)$ and $\tilde{K}_1(E) \subset K_1(E)$. If $z = (1+\omega)/(1-\omega)$, where $z \in \Pi$, $\omega \in E$, then one-to-one correspondence between classes $\tilde{K}_1(\Pi)$ and $\tilde{K}_1(E)$ is established by the formula

$$F(z) = 2g(\omega). \tag{3.1}$$

Dependence between the derivatives of these functions is defined by the formula

$$\frac{F^{(n)}(z)}{n!} = \frac{(\omega-1)^{n+1}}{2^{n-1}} \sum_{k=0}^{n-1} C_{n-1}^k \frac{g^{(k+1)}(\omega)}{(k+1)!} (\omega-1)^k, \ldots n = 1, 2, \ldots, \tag{3.2}$$

where C_{n-1}^k are binomial coefficients ([6]). For any $g(\omega) \in \tilde{K}_1(E)$ the estimates

$$\frac{|g^{(m)}(\omega)|}{m!} \leq \frac{m+|\omega|}{(1-|\omega|)^{m+2}}, \quad m = 0, 1, 2, \ldots \tag{3.3}$$

$$\frac{1-|\omega|}{(1+|\omega|)^3} \leq |g'(\omega)| \leq \frac{1+|\omega|}{(1-|\omega|)^3} \tag{3.4}$$

are valid (see [1, 2]). Using relationships (3.1) – (3.4) for the function $F(z) \in \tilde{K}_1(\mathit{\Pi})$ we obtain the following estimations:

$$\frac{(1-|\omega|)\,|1-\omega|^2}{(1+|\omega|)^3} \leq |F'(z)| \leq \frac{(1+|\omega|)\,|1-\omega|^2}{(1-|\omega|)^3}, \tag{3.5}$$

$$\frac{\left|F^{(n)}(z)\right|}{n!} \leq \frac{|1-\omega|^{n+1}}{2^{n-1}} \sum_{k=0}^{n-1} C_{n-1}^k \frac{\left|g^{(k+1)}(\omega)\right|}{(k+1)!} |1-\omega|^k, \quad n = 1, 2, \ldots \tag{3.6}$$

4. Proof of Theorem 1

First, let us consider the case, when $m = 1$ and then $k = n \geq 6$. Let us suppose that a certain function $F_1(z)$ belongs simultaneously to two classes $Q_1(\mathit{\Pi})$ and $Q_n(\mathit{\Pi})$, where $n \geq 6$. Since $F_1(z) \in Q_1(\mathit{\Pi})$, then according to Lemma 1 we have $F_1(z) \in K_1(\mathit{\Pi})$. But then due to Lemma 2 we obtain

$$F(z) = \frac{F_1(z) - F_1(1)}{F_1'(1)} \in \tilde{K}_1(\mathit{\Pi}).$$

Further, since $F_1(z) \in Q_n(\mathit{\Pi})$, then from Lemma 3 we have $F_1^{(n-1)}(z) \in K_1(\mathit{\Pi})$. By using Lemma 2, we obtain that $F^{(n-1)}(z) \in K_1(\mathit{\Pi})$ and

$$\mathit{\Psi}(z) = \frac{F^{(n-1)}(z) - F^{(n-1)}(1)}{F^{(n)}(1)} \in \tilde{K}_1(\mathit{\Pi}).$$

Using (3.4), we get

$$\frac{(1-|\omega|)\,|1-\omega|^2}{(1+|\omega|)^3} \leq |\mathit{\Psi}'(z)| = \left|\frac{F^{(n)}(z)}{F^{(n)}(1)}\right| \leq \frac{(1+|\omega|)\,|1-\omega|^2}{(1-|\omega|)^3}. \tag{4.1}$$

Taking into account (3.6), the estimations (4.1) can be written in the following form

$$\frac{(1-|\omega|)\,|1-\omega|^2}{(1+|\omega|)^3} \leq |\mathit{\Psi}_1'(z)| = \left|\frac{F^{(n)}(z)}{F^{(n)}(1)}\right|$$

$$\leq \frac{n!\,|1-\omega|^{n+1}}{\left|F^{(n)}(1)\right| 2^{n-1}} \sum_{k=0}^{n-1} C_{n-1}^k \frac{\left|g^{(k+1)}(\omega)\right|}{(k+1)!} |1-\omega|^k.$$

Using estimation (3.3), we get the dual inequality

$$\frac{(1-|\omega|)\,|1-\omega|^2}{(1+|\omega|)^3} \leq \left|\frac{F^{(n)}(z)}{F^{(n)}(1)}\right| \leq \frac{n!\,|1-\omega|^{n+1}}{\left|F^{(n)}(1)\right| 2^{n-1}} \sum_{k=0}^{n-1} C_{n-1}^{k} \frac{k+1+|\omega|}{(1-|\omega|)^{k+3}} |1-\omega|^{k}.$$

From here it follows that

$$\frac{(1-|\omega|)\,|1-\omega|^2}{(1+|\omega|)^3} \leq \frac{n!\,|1-\omega|^{n+1}}{\left|F^{(n)}(1)\right| 2^{n-1}} \sum_{k=0}^{n-1} C_{n-1}^{k} \frac{k+1+|\omega|}{(1-|\omega|)^{k+3}} |1-\omega|^{k}. \qquad (4.2)$$

Let $\omega = t$ in (4.2), where $0 < t < 1$. Then

$$2^{n-1}\frac{\left|F^{(n)}(1)\right|}{n!} \leq (1+t)^3 (1-t)^5 \sum_{k=0}^{n-1} C_{n-1}^{k} (k+1+t).$$

Taking into account Lemma 4 we obtain

$$\frac{\left|F^{(n)}(1)\right|}{n!} \leq \frac{(1+t)^3 (n+1+2t)(1-t)^{n-5}}{2}.$$

Since $F^{(n)}(z) \neq 0$ and $n \geq 6$, then the latter inequality cannot be fulfilled when values of t are close enough to 1. The obtained contradiction proves theorem for the first case, when $m = 1$. Let now $m \geq 2$. If $F(z) \in Q_m(\Pi)$ and $F(z) \in Q_k(\Pi)$, then, designating $\Phi(z) = F^{(m-1)}(z)$ and using Lemma 3, we will obtain that $\Phi(z) \in Q_1(\Pi)$ and $\Phi(z) \in Q_n(\Pi)$. As a result we again have the first case and Theorem 1 is proven.

Remark. A question about existence of the univalent in the half-plane functions, which belong to class $Q_5(\Pi)$, remains open.

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