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EXPLOSIVE INSTABILITY IN RIMMING FLOWS

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Abstract. The linear stability of a thin film of viscous fluid on the inside of a cylinder with horizontal axis, rotating about this axis is examined in this paper. Both axial and azimuthal components of the hydrostatic pressure gradient are taken into account, which yield solutions that collapse in both dimensions. Despite the existence of these explosive instabilities, all solutions with harmonic dependence on the axial variable and time are neutrally stable. This type of instability has been described in previous papers. However, no actual solution to describe the movement of the film of liquid through the cylinder has been presented. This paper will rectify this and examine the properties of such a solution.

Key words: rimming flow, harmonic disturbances, explosive instability

1. Introduction

The flow of a thin film of liquid entrained on the inside of a rotating horizontal cylinder is commonly termed as rimming flow (see Figure 1). Recent developments in rimming flow have progressed with the discovery of linear equations with very unusual properties. In [4], and later in [2], the concept of 'explosive' instability was introduced and these properties are described. Initialy in [4] an asymptotic solution for an hydrodynamic system is derived by considering two orders of Moffat's [5] lubrication approximation. An asymptotic solution for describing three-dimensional motion of the film is derived in [2]. Both papers subsequently analysed the stability of the steady-state distribution, with the resulting linearised problem indicating infinitely many neutrally stable harmonic disturbances. However [2, 4] also found the presence of non-harmonic solutions that developed singularities in a finite time. This alternative source of instability, whereby each term of the series was bounded but the series as a whole diverged, was interpreted as an 'exploding' instability. In [2, 4] the collapse (explosion) of the film occurred in the azimuthal and axial directions, respectively.

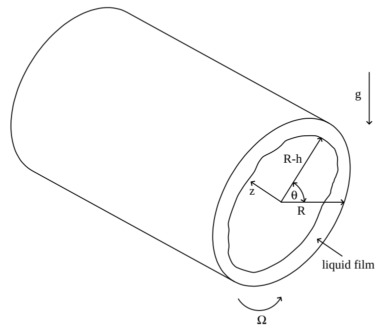


Figure 1. Formulation of the problem: liquid film inside a rotating horizontal cylinder.

This paper sets out to present a model where both the axial and azimuthal components of the hydrostatic pressure gradient are taken into account. The resulting solution will produce infinitely many stable normal modes (with harmonic dependence on the axial variable and time), along with non-harmonic disturbances that will collapse in both the axial and azimuthal directions.

2. Formulation of the Problem

2.1. The governing equation

Consider a thin film of incompressible liquid on the inside of a cylinder of radius R with a horizontal axis. The cylinder is rotating about this axis with constant angular velocity Ω (see Figure 1). Cylindrical coordinates (r, θ, z) are used. h, the thickness of the film, depends on the azimuthal angle θ, axial coordinate z, and time t. The following non-dimensional variables are used:

$$\theta = \hat{\theta}, \qquad z = \sqrt{\varepsilon}\frac{\hat{z}}{R}, \qquad t = \Omega\hat{t}, \qquad \lambda = \frac{\hat{\lambda}}{\varepsilon R}\left(1 - \frac{\hat{\lambda}}{2R}\right),$$

where g, υ are parameters describing the acceleration due to gravity and the kinematic viscosity of the fluid, respectively. $\varepsilon = \sqrt{\dfrac{\upsilon\Omega}{gR}}$ is essentially the non-dimensional thickness of the film.

The combined effect of the axial and azimuthal components of the hydrostatic pressure gradient are examined in this paper. To derive the most general model, the former is treated as a leading-order effect, whereas the latter is treated as a perturbation. The approach to modeling these effects is based on the lubrication approximation. As a result, the following equation can be derived:

$$\frac{\partial\lambda}{\partial t}+\frac{\partial}{\partial\theta}\Big\{\lambda-\frac{1}{3}\lambda^3\cos\theta+\varepsilon\big[\frac{1}{3}\lambda^3\frac{\partial\lambda}{\partial\theta}\sin\theta-\frac{1}{2}\lambda^3\big(\frac{\partial\lambda}{\partial z}\big)^2\cos\theta-\frac{5}{24}\lambda^4\frac{\partial^2\lambda}{\partial z^2}\cos\theta\big]\Big\}$$
$$+\frac{\partial}{\partial z}\left\{\frac{1}{3}\lambda^3\frac{\partial\lambda}{\partial z}\sin\theta+\varepsilon\left[\frac{3}{5}\lambda^5\frac{\partial^3\lambda}{\partial z^3}\sin\theta+4\lambda^4\frac{\partial\lambda}{\partial z}\frac{\partial^2\lambda}{\partial z^2}\sin\theta+\frac{7}{3}\lambda^3\left(\frac{\partial\lambda}{\partial z}\right)^3\sin\theta\right.\right.$$
$$\left.\left.-\frac{19}{24}\lambda^4\frac{\partial^2\lambda}{\partial\theta\partial z}\cos\theta-\frac{11}{6}\lambda^3\frac{\partial\lambda}{\partial\theta}\frac{\partial\lambda}{\partial z}\cos\theta+\frac{35}{24}\lambda^4\frac{\partial\lambda}{\partial z}\sin\theta\right]\right\}=0. \quad (2.1)$$

A similar method for the derivation of (2.1) can be found in [2, 4].

2.2. Steady states and disturbances

Let the solution of equation (2.1) be independent of t and z, i.e. $\lambda(\theta,z,t)=\bar\lambda(\theta)$. Then, (2.1) yields

$$\bar\lambda-\tfrac{1}{3}\bar\lambda^3\cos\theta+\tfrac{1}{3}\varepsilon\bar\lambda^3\frac{\partial\bar\lambda}{\partial\theta}\sin\theta=q, \quad (2.2)$$

where q is a constant of integration (physically, q is the non-dimensional mass flux). In order to examine the properties of $\bar\lambda$ for stability, assume that

$$\lambda(\theta,z,t)=\bar\lambda\,(\theta)+\lambda'(\theta,z,t), \quad (2.3)$$

where λ' represents a small disturbance. Substitute (2.3) into (2.2) and omit the nonlinear terms, to obtain (primes are omitted in notation)

$$\frac{\partial\lambda}{\partial t}+\frac{\partial}{\partial\theta}\left\{\lambda\left(1-\bar\lambda^2\cos\theta+\varepsilon\bar\lambda^2\frac{\partial\bar\lambda}{\partial\theta}\right)+\varepsilon\left[\frac{1}{3}\bar\lambda^3\frac{\partial\lambda}{\partial\theta}\sin\theta-\frac{5}{24}\bar\lambda^4\frac{\partial^2\lambda}{\partial z^2}\cos\theta\right]\right\}$$
$$+\frac{\partial}{\partial z}\left\{\frac{1}{3}\bar\lambda^3\frac{\partial\lambda}{\partial z}\sin\theta+\varepsilon\left[\left(\frac{3}{5}\bar\lambda^5\frac{\partial^3\lambda}{\partial z^3}+\frac{35}{24}\bar\lambda^4\frac{\partial\lambda}{\partial z}\right)\sin\theta\right.\right.$$
$$\left.\left.-\left(\frac{19}{24}\bar\lambda^4\frac{\partial^2\lambda}{\partial\theta\partial z}+\frac{11}{6}\bar\lambda^3\frac{\partial\bar\lambda}{\partial\theta}\frac{\partial\lambda}{\partial z}\right)\cos\theta\right]\right\}=0. \quad (2.4)$$

The main difficulty associated with this equation is that the explicit form of its coefficients is unknown. The only exception is the limit of small flux, $q\ll 1$, in which case (2.2) admits an explicit asymptotic asymptotic solution,

$$\bar\lambda=q+\tfrac{1}{3}q^3\cos\theta+O(q^5),\quad \text{if } q\ll 1. \quad (2.5)$$

Substitute (2.5) into (2.4) and assume, for simplicity, that $\dfrac{\varepsilon q^3}{\Delta\theta}\gg q^4$ (where $\Delta\theta$ is the characteristic azimuthal scale of the solution). Omit the $O(q^4)$ terms and smaller to obtain

$$\frac{\partial\lambda}{\partial t}+\frac{\partial}{\partial\theta}\left[\left(1-q^2\cos\theta\right)\lambda+\frac{1}{3}\varepsilon q^3\sin\theta\frac{\partial\lambda}{\partial\theta}\right]+\frac{\partial}{\partial z}\left(\frac{1}{3}q^3\sin\theta\frac{\partial\lambda}{\partial z}\right)=0. \quad (2.6)$$

Equation (2.6) describes the azimuthal (θ) propagation of disturbances and their diffusion in both θ- and z-directions. Accordingly, the factor $\left(1 - q^2 \cos\theta\right)$ is the propagation speed, whereas $-\frac{1}{3}\varepsilon q^3 \sin\theta$ and $-\frac{1}{3}q^3 \sin\theta$ are the effective diffusion coefficients in the θ- and z-directions, respectively. Note, in the lower half of the cylinder ($-\pi < \theta < 0$), the diffusion is positive, whereas in the upper half ($0 < \theta < \pi$), the coefficients are both negative (because of 'inverse' gravity).

3. Harmonic Disturbances

In this section, the normal modes (solutions with harmonic dependence on the axial variable and time) will be examined, i.e.

$$\lambda(\theta, z, t) = \phi(\theta)\, \mathrm{e}^{\mathrm{i}(kz - \omega t)}, \tag{3.1}$$

where ω is the frequency and k is the axial wave number. Substitution of (3.1) into (2.6) yields

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\left(1 - q^2 \cos\theta\right)\phi + \epsilon \sin\theta \frac{\mathrm{d}\phi}{\mathrm{d}\theta}\right] - \left(\mathrm{i}\omega + \kappa^2 \sin\theta\right)\phi = 0, \tag{3.2}$$

where $\epsilon = \dfrac{\varepsilon q^3}{3}$, $\kappa^2 = \dfrac{k^2 q^3}{3}$.

Equation (3.2) together with the periodicity condition,

$$\phi(\theta + 2\pi) = \phi(\theta), \tag{3.3}$$

form an eigenvalue problem, where $\phi(\theta)$ is the eigenfunction and ω is the eigenvalue. If Im $\omega > 0$ the film is unstable.

Equations (3.2)–(3.3) can be solved asymptotically, using a WKB-type method based on the smallness of ϵ. [3] shows, using this method, all disturbances are neutrally stable. This paper will introduce a numerical method to corroborate the findings of [3].

3.1. Numerical Method

The solution of (3.2)–(3.3) can be represented by its complex Fourier series

$$\phi(\theta) = \sum_{k=-\infty}^{\infty} \phi_k \mathrm{e}^{\mathrm{i}k\theta}, \tag{3.4}$$

where ϕ_k are the Fourier coefficients. Substitution of (3.4) into (3.2) together with routine algebra yields

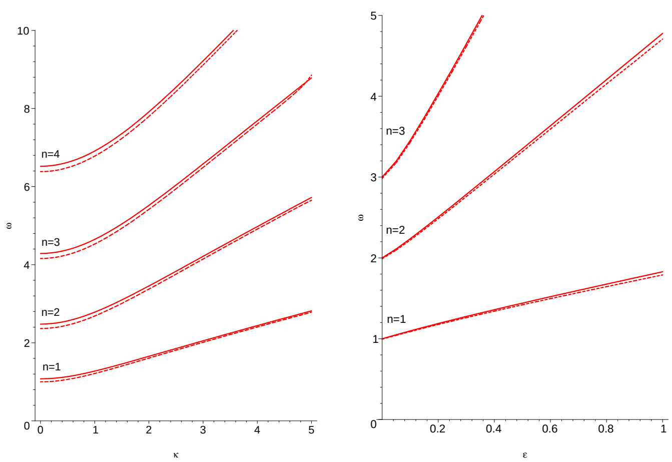
$$\sum_{l=-\infty}^{\infty} A_{k,l}\phi_l = \omega\phi_k,$$

where

$$A_{k,l} = k\delta_{k,l} + \frac{k\left[\epsilon\,(k-1) - q^2\right] + \kappa^2}{2}\delta_{k-1,l} - \frac{k\left[\epsilon\,(k+1) + q^2\right] + \kappa^2}{2}\delta_{k+1,l},$$

and $\delta_{k,l}$ is the Kronecker delta. Thus problem (3.2)–(3.3) is reduced to an eigenvalue problem of an infinite tri-diagonal matrix $A_{k,l}$. In practice $A_{k,l}$ is truncated at a large but finite size where its eigenvalues are computed using a suitable numerical algorithm, e.g. MATLAB's *eig* function for sparse matrices.

The results obtained through the WKB method from [3] can be compared to the direct numerical solution of problem (3.2)–(3.3). For all physically relevant values of parameters (ϵ, $q^2 \leq 0.1$) the difference between the asymptotic and exact solutions is hardly visible. Figure 2(a) shows the results for $\epsilon = q^2 = 0.3$, in which case the agreement between the two solutions is still very good. Figure 2(b) shows that the accuracy of the WKB method improves with growing mode number. The third mode is indistinguishable from the exact solution.



(a) (b)

Figure 2. The dotted line shows the exact numerical solution of (3.2)–(3.3), the solid line represents the solution achieved using the WKB method in [3] of (a) the eigenvalues ω, with $\kappa \in [0,5]$, $\epsilon = q^2 = 0.3$ for the first four modes, (b) the eigenvalues ω, with $\epsilon \in [0,1]$, $\kappa = 1$, $q^2 = 0.1$ for the first three modes.

4. Exploding Solutions

Despite all the normal modes of (3.2)–(3.3) being stable, equation (2.6) admits an exploding solution, which develops a singularity in a finite time. Although, this type of instability (with a collapse in both the axial and azimuthal directions) has been described in [3], no actual expression has been presented (see [3] for the method in yielding this exploding solution). Thus, the exploding solution has the following form:

$$\lambda\Big(\theta, z, t\Big) = \frac{A(t)}{\sqrt{\left(\theta - t\right)^2 + \varepsilon z^2}} e^{-\frac{(\theta-t)^2+\varepsilon z^2}{4W^2(t)}} \mathrm{M}\Big(\frac{\alpha-1}{2}, 0, \frac{\left(\theta-t\right)^2+\varepsilon z^2}{2W^2(t)}\Big), \quad (4.1)$$

where M is the *WhittakerM*-function (see [1]). Function (4.1) describes a narrow pulse advancing along the inner surface of the cylinder in a anti-clockwise direction with width and amplitude given by

$$W(t) = \sqrt{W_0^2 - 4\varepsilon \sin^2\left(\frac{1}{2}t\right)}, \qquad A\left(t\right) = A_0 W^{1-\alpha}(t), \qquad (4.2)$$

where α is a constant that was introduced in the course of a separation of variables. It is implied that the pulse's initial width and amplitude are $W_0,\ A_0$.

(4.1)-(4.2) together show that the evolution of the pulse depends on whether or not the initial width W_0 exceeds a threshold value of $2\sqrt{\varepsilon}$.

- If $W_0 > 2\sqrt{\varepsilon}$, the solution is smooth at all times. Between $t = 0$ and $t = \pi$, the pulse is traveling through the upper half of the cylinder, where the diffusivity is *negative.* Accordingly, the width of the pulse is decreasing and the amplitude is growing. At $t = \pi$, the pulse enters the region of *positive* diffusivity, and by the time it reaches the starting point ($t = 2\pi$), it restores its initial parameters. This cycle repeats itself indefinitely.
- If $W_0 \leq 2\sqrt{\varepsilon}$, it follows

$$W\left(t\right) \to 0, \quad A\left(t\right) \to \infty \quad \text{as} \ \ t \to 2\arcsin\left(\frac{W_0}{2\sqrt{\varepsilon}}\right).$$

  Thus, if the pulse is sufficiently narrow initially, it blows up due to the effect of 'anti-diffusivity' before it leaves the upper half of the cylinder. Observe that α determines how quickly the amplitude of the pulse tends to infinity as $W \to 0$ – accordingly, α is referred to as the explosion rate.

5. Concluding Remarks

This paper firstly presents a model of rimming flows (i.e. equation (2.1)), which includes both axial and azimuthal components of the pressure gradient. All solutions with harmonic dependence on the axial variable and time (normal modes) are found to be neutrally stable to the second order. This result is verified numerically in Section 3.

However, an additional linear stability analysis yields a system that admits solutions which develop singularities in a finite time. These solutions are shown to collapse (explode) in both the axial and azimuthal directions and are represented formally by (4.1)–(4.2). [3] conjectures that this exploding solution describes the early stage of a drop formation. For example, imagine a short-scale perturbation on the surface of the film. When the rotation of the cylinder turns the perturbation 'upside down', gravity starts increasing its amplitude and/or shortening its size. Once the perturbation is sufficiently large and narrow, a drop of fluid should detach itself from the film and fall down.

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