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THE NONLINEAR AND NONLOCAL INTEGRABLE SINE-GORDON EQUATION

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Abstract. A new type of the nonlocal sine-Gordon equation with the generalized interaction term is suggested. Its limit cases, symmetries and exact analytical solutions are obtained. This type of the nonlocal sine-Gordon equation is shown to possess one-, two- and N-solitonic solutions which are a nonlocal deformation of the corresponding classical solutions of the sine-Gordon equation.

1. Introduction

The sine-Gordon equation (SGE)

$$\phi_{tt} - a\phi_{xx} = b\,\sin(\lambda\phi) \tag{1.1}$$

is one of the basic nonlinear equations both in mathematics and modern physics. In mathematics its appears as an equation for the surfaces of constant negative curvature ($a = \lambda = -b = 1$) and was known even to F. Minding and E. Beltrami. The physical applications are related with the description of dislocations in solid state physics [16], motion of Bloch magnetic walls in magnetic crystals [14], magnetic flux propagation in superconductors [10] and so on [12]. In these applications the SGE gives the simplest nonlinear description of phenomena under consideration. More adequate models correspond to SGE generalizations (1.1).

All known nonlocal generalizations of SGE could be divided into two groups: 1) where the kinetic or 2) the dynamic term is under nonlocal generalization. To the first group belong various generalizations where the local operator $\partial_{xx}\phi$ is replaced by the integro-differential operator $L[\phi]$ [1]:

$$\phi_{tt} - L[\phi] = b\,\sin(\lambda\phi)\,. \tag{1.2}$$

In particular, various interesting examples of nonlocal Josephson electrodynamics belong to the family of the evolution equation (1.2). These examples were introduced in [18]-[7], in which one of the basic model equations is

$$\phi_{tt} - \hat{H}\phi_x + \sin\phi = 0\,, \tag{1.3}$$

where $\hat{H}$ is the Hilbert transform. The evolution equation (1.3) was an object of study in a series of papers [17, 19],[8]-[5].

To the first group belongs also the nonlocal generalization of SGE proposed in [2]:

$$\phi_{tt} - {}^R D_x^\alpha \phi + \sin\phi = 0\,, \tag{1.4}$$

where ${}^R D_x^\alpha$ is the Riesz partial fractional derivative. For this equation, a family of breather-like solutions (i.e. solutions that are localized in space and periodic in time) has been found numerically, and it has been shown that these entities are quite robust and can be generated in the course of evolution of initial states of a rather different shape.

Another type of nonlocal generalization of SGE was proposed in [11, 21]:

$$\phi_{tt} - \phi_{xx} = -2\cos\left[\frac{\phi(x,t)}{2}\right]\int f(x-y)\,\sin\left[\frac{\phi(y,t)}{2}\right]dy\,, \tag{1.5}$$

where $f(x) = 1/(x^4 + \sigma^4)$ or Gauss-type. It is shown that small amplitude solitons of the nonlocal SGE can create coupled states. The effect is due to a change of the dispersion originated by nonlocal nonlinearity. The evolution equation (1.5) in the general case could be generalized in the form

$$\phi_{tt} - \phi_{xx} = F[\phi]\,, \tag{1.6}$$

where $F[\phi]$ is a function of $\phi(x,t)$.

In the current contribution, a new type of nonlocal SGE is suggested. Exact analytical solutions of this equation and its Lagrangian are considered.

2. Nonlocal Generalization of Sine-Gordon Equation

Let us consider a special type of the nonlocal SGE (NSGE):

$$\phi_{tt} - a\phi_{xx} = b\,{}^R D_x^{-\alpha}\sin\left(\lambda\,{}^R D_x^\alpha \phi\right), \tag{2.1}$$

where a, b and λ are constants and ${}^R D_x^\alpha$ means a space fractional Riesz derivative of the order α (see Appendix). This equation belongs to the second group of the possible nonlocal generalizations of SGE (1.6), where the term of potential interaction is modified.

At first sight this equation looks very complicated, but actually it is an equivalent transformation of the interaction term. Indeed, in the case of linear dependence this term does not change.

In the case of small values of the parameter α, the infinitesimal form of equation (2.1) is as follows:

$$\phi_{tt} - a\phi_{xx} = b\sin\lambda\phi + \alpha L[\phi]\,, \tag{2.2}$$

where $L[\phi]$ is a local perturbation of the classical SGE. When $\alpha \to 0$ the NSGE turns into the ordinary SGE (1.1).

In the case of small amplitudes $|\lambda^R D_x^\alpha \phi| \ll 1$, the NSGE turns into the linear Klein–Gordon equation with the "mass" term $\lambda b\phi$.

If $\phi(x,t)$ is a solution of the SGE, then the function

$$\phi_1(x,t) = \frac{2\pi n}{\lambda} \pm \phi(C_1 \pm x, C_2 \pm t)\,, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{2.3}$$

where C_1, C_2 are arbitrary constants, is also an exact solution of SGE. The signs in expression (2.3) could be chosen arbitrarily. Unfortunately, this does not hold for NSGE solutions, but would be useful for generating new solutions of NSGE by the known solution of SGE.

2.1. The Lagrangian

It could be verified that the NSGE (2.1) has the Lagrangian form:

$$L = \int_{-\infty}^{+\infty} \left\{ \frac{1}{2} \left[({}^R D_x^\alpha \phi_t)^2 - ({}^R D_x^{1+\alpha} \phi)^2 \right] + \frac{b}{\lambda} \left[1 - \cos\left(\lambda^R D_x^\alpha \phi\right)\right] \right\} dx. \tag{2.4}$$

Thus, the equation of motion (2.1) can be derived by using the modified Noether theorem. For instance, the energy-momentum tensor T_{ik} in the Minkowsky metric η_{ik}:

$$T_{ik} = ({}^R D_x^\alpha \phi_i)({}^R D_x^\alpha \phi^l) g_{lk} - g_{ik}\mathcal{L}\,, \tag{2.5}$$

where $\mathcal{L}$ is the Lagrangian density in the expression (2.4).

2.2. The travelling wave solution

The NSGE has the travelling wave solution – a nonlocal generalization of one-solitonic solution.

a) Let $b\lambda(\mu^2 - ak^2) > 0$, then

$$\phi(x,t) = \frac{4}{\lambda}\ {}^R D_x^{-\alpha} \mathrm{arctg} \left\{ \exp\left[\pm \frac{b\lambda(kx + \mu t + \theta_0)}{\sqrt{b\lambda(\mu^2 - ak^2)}} \right] \right\}, \tag{2.6}$$

where k, μ, θ_0 are arbitrary constants.

b) Let $b\lambda(\mu^2 - ak^2) < 0$, then

$$\phi(x,t) = \frac{4}{\lambda}\ {}^R D_x^{-\alpha} \mathrm{arctg} \left\{ \exp\left[\pm \frac{b\lambda(kx + \mu t + \theta_0)}{\sqrt{b\lambda(ak^2 - \mu^2)}} \right] \right\}, \tag{2.7}$$

where k, μ, θ_0 like above are arbitrary constants.

3. Conclusions

Thus, the NSGE, like the ordinary SGE, has a Lagrangian form (2.4), one-solitonic solutions (2.6). Despite the nonlocal nature of the interaction term in the evolution equation, this model possesses nonlocal deformations of localized solutions.

The asymptotic form has slowly falling tails $\phi(x) \sim x^{\alpha}$ which converge to zero at $\alpha < 0$, as follows from explicit expressions of the solutions. At the same time the total value of the momenta

$$I[\phi] = \int_{-\infty}^{+\infty} \phi(x,0)\,dx$$

diverges for any $\alpha > -1$. This means a nonlocal distribution of the momenta, energy and related magnitudes.

From the asymptotic and infinitesimal form of NDGE (2.1) the corresponding dispersion relations follow:

$$\omega^2 - ak^2 = \lambda b, \qquad \omega^2 - ak^2 = W(k)\,, \tag{3.1}$$

where $W(k)$ corresponds to the Fourier transform for the linearized part of the $b\sin\lambda\phi + \alpha L[\phi]$ according to equation (2.2) and which are the Klein–Gordon and *sine*-Gordon modified dispersion relations.

The NSGE could be obtained from the discretized version of the evolution equation:

$$y_n'' = \frac{a}{s^2}(y_{n+1} - 2y_n + y_{n-1}) + \lambda y_n \cos\left[\lambda s^{-\alpha}(y_{n+1} - y_n)\right], \tag{3.2}$$

where s is the length of the space step displacement.

The variety of the physical origination of SGE (1.1) allows us to apply the obtained solutions not only to the Josephson effect [6-11], but also to dislocations evolution in the modified Frenkel–Kontorova model [1], magnetic crystals [2], semiconductors [3] and so on.

Note here one important property. The continuation of the parameter $\alpha \in [0;2]$ does not mean a continuations transition of one evolution equation to another. Let us have an evolution equation in the form

$$\phi_{tt} - a\phi_{xx} = N_\alpha[\phi]\,, \tag{3.3}$$

where $N_\alpha[\phi]$ means the nonlocal operator on $\phi(x,t)$, and α is the parameter of nonlocality. The transformation of the operator $N_\alpha[\phi]$ for $\alpha \in [0;2]$ induces the transformation of the automorphism groups G_0 and G_2 for the corresponding local evolution equations:

$$\begin{array}{ccccc}
N_0[\phi] & \xrightarrow{\alpha} & D^\alpha & \xrightarrow{\alpha} & N_2[\phi] \\
{\scriptstyle Aut}\downarrow & & \downarrow & & \downarrow{\scriptstyle Aut} \\
G_0 & \xrightarrow{\alpha} & T^\alpha & \xrightarrow{\alpha} & G_2
\end{array}$$

where, in the general case, the operator of the fractional derivative D^α induces an action on the group of translation operators T^α.

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