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A PRIORI BOUNDS ON THE SOLUTIONS OF SINGULARLY PERTURBED ELLIPTIC DIFFUSION-CONVECTION-REACTION PROBLEMS [1]

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Abstract. A priori parameter explicit bounds on the derivatives of the solution of a two parameter singularly perturbed elliptic problem in two space dimensions are presented. These bounds are used to establish parameter uniform error bounds for a numerical method consisting of upwinding on a tensor product of two piecewise uniform meshes.
Key words: elliptic equation, two parameters, a priori bounds

1. Statement of Problem

Consider the following class of singularly perturbed elliptic problems posed on the unit square $\Omega = (0,1)^2$,

$$L_{\varepsilon,\mu}u = \varepsilon(u_{xx} + u_{yy}) + \mu(a_1 u_x + a_2 u_y) - bu = f \quad \text{in } \Omega, \tag{1.1a}$$

$$u|_{\Gamma_B} = s_1(x), \quad u|_{\Gamma_T} = s_2(x), \quad u|_{\Gamma_L} = q_1(y), \quad u|_{\Gamma_R} = q_2(y), \tag{1.1b}$$

$$s_1(0) = q_1(0),\ s_2(0) = q_1(1),\ s_1(1) = q_2(0),\ s_2(1) = q_2(1), \tag{1.1c}$$

$$a_1(x,y) \geq \alpha_1 > 0,\ a_2(x,y) \geq \alpha_2 > 0,\ b(x,y) \geq 2\beta > 0, \tag{1.1d}$$

where $\Gamma_B, \Gamma_T, \Gamma_L$ and Γ_R are the edges of the boundary $\partial\Omega$ defined by

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$$\Gamma_B = \{(x,0)|0 \le x \le 1\}, \quad \Gamma_T = \{(x,1)|0 \le x \le 1\},$$

$$\Gamma_L = \{(0,y)|0 \le y \le 1\}, \quad \Gamma_R = \{(1,y)|0 \le y \le 1\}.$$

Throughout this paper, we assume sufficient regularity and compatibility on the data so that the solution and its components are sufficiently smooth for the following analysis to be valid. With respect to regularity assume that $a_1, a_2, b, f \in C^{n,\alpha}(D), s_1, s_2, q_1, q_2 \in C^m(J)$, where D, J are open sets such that $\overline{\Omega} \subset D,\ [0,1] \subset J$ and n, m are sufficiently large for our analysis. In this paper, the norm $\|v\|_R = \max_{\vec{x}\in R} |v(\vec{x})|$ is the maximum pointwise norm.

Lemma 1. *The solution u of* (1.1)*, satisfies the following bounds*

$$||u|| \le ||s||_{\Gamma_B\cup\Gamma_T} + ||q||_{\Gamma_L\cup\Gamma_R} + \frac{1}{\beta}||f||$$

and for $1 \le k+m \le 3$,

$$\left|\left|\frac{\partial^{k+m}u}{\partial x^k \partial y^m}\right|\right| \le \frac{C}{(\sqrt{\varepsilon})^{k+m}}\Big(1 + \Big(\frac{\mu}{\sqrt{\varepsilon}}\Big)^{k+m}\Big),$$

where C depends on the coefficients a_1, a_2, b , the boundary data s_1, s_2, q_1, q_2, the inhomogeneous term f and their derivatives.

Note that the differential equation (1.1a) contains two singular perturbation parameters $0 < \varepsilon \le \varepsilon_0 = \mathcal{O}(1)$ and $0 \le \mu \le 1$. The analysis for this two-parameter problem naturally splits into two cases, $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$ and $\mu^2 \ge \frac{\gamma\varepsilon}{\alpha}$. In the case of $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$, the analysis is similar to that in the case of $\mu = 0$ and boundary layers of width $\mathcal{O}(\sqrt{\varepsilon})$ appear in the neighbourhood of all four edges. For the case of $\mu^2 \ge \frac{\gamma\varepsilon}{\alpha}$ the analysis is more intricate and boundary layers of width $\mathcal{O}(\frac{\varepsilon}{\mu})$ appear in the neighbourhood of the edges $x = 0$, $y = 0$ and boundary layers of width $\mathcal{O}(\mu)$ appear in the neighbourhood of $x = 1, y = 1$.

In this paper, we confine the discussion to the case of $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$ and throughout we assume that

$$\mu^2 \le \frac{\gamma\varepsilon}{\alpha}, \quad \gamma = \min\left\{\frac{b}{2a_1}, \frac{b}{2a_2}\right\}. \tag{1.1e}$$

2. Regular Component

In order to obtain more informative parameter explicit error bounds on the derivatives of the solution of (1.1), the solution is decomposed into a sum of regular and layer components. The extension idea from [3] is used to define the regular solution, which avoids imposing overly artificial compatibility conditions. We show that there exists a function v such that $L_{\varepsilon,\mu}v = f$ and when its boundary conditions are chosen appropriately, the function v and its derivatives up to second order are bounded independently of the small parameters.

Define the zero order differential operator L_0 as follows

$$L_0 z = -bz.$$

Consider the extended domain $\Omega^* = (-d, 1+d) \times (-d, 1+d) \supset \overline{\Omega}$, $d > 0$. The extended differential operators $L^*_{\varepsilon,\mu}$ and L^*_0 coincide with the operators $L_{\varepsilon,\mu}$ and L_0 respectively on Ω. We also define smooth extensions a^*_1, a^*_2, b^* and f^* of the functions a_1, a_2, b and f to Ω^*.

Consider the differential equation $L^*_{\varepsilon,\mu} v^* = f^*$ on Ω^* and decompose v^* as follows

$$v^*(x,y) = v^*_0(x,y) + \sqrt{\varepsilon} v^*_1(x,y) + \varepsilon v^*_2(x,y),$$

where

$$L^*_0 v^*_0 = f^*, \quad \sqrt{\varepsilon} L^*_0 v^*_1 = (L^*_0 - L^*_{\varepsilon,\mu}) v^*_0,$$
$$\varepsilon L^*_{\varepsilon,\mu} v^*_2 = \sqrt{\varepsilon}(L^*_0 - L^*_{\varepsilon,\mu}) v^*_1, \quad v^*_2|_{\partial\Omega^*} = 0.$$

Note that v^*_0 and v^*_1 satisfy zero order differential equations and hence there are no issues of compatibility. The term v^*_2 is the solution of an elliptic problem on the extended domain Ω^*. The extensions are taken so that the function $g^* \equiv (L^*_0 - L^*_{\varepsilon,\mu}) v^*_1$ is zero at the four corners of the extended domain and $g^* \in C^{1,\alpha}(\bar{\Omega}^*)$. In this way the term $v_2 \in C^{3,\alpha}(\bar{\Omega}^*)$ is sufficiently regular for our purposes [2].

Define the regular component v to be the solution of the elliptic problem

$$\begin{cases} L_{\varepsilon,\mu} v = f, \ (x,y) \in \Omega, \\ v = v^*, \ \ (x,y) \in \partial\Omega. \end{cases}$$

Assuming sufficient smoothness of the coefficients, we can establish the following bounds on the first three derivatives of the regular component v

$$\left\| \frac{\partial^{k+m} v}{\partial x^k \partial y^m} \right\| \le C\big(1 + \varepsilon^{\frac{2-k-m}{2}}\big), \ 0 \le k+m \le 3, \quad \text{if} \quad \mu^2 \le \frac{\gamma\varepsilon}{\alpha}.$$

3. Layer Components

Associated with the left edge Γ_L, we define a boundary layer function w_L. Consider the extended domain $\Omega^{**} = (0,1) \times (-d, 1+d)$, $0.5 > d > 0$. We define w^*_L to be the solution of

$$\begin{cases} L^{**}_{\varepsilon,\mu} w^*_L = 0, \quad (x,y) \in \Omega^{**}, \\ w^*_L|_{\Gamma_L} = u - v, \quad w^*_L(1,y) = 0, \quad y \in [-d, 1+d], \\ w^*_L(x,-d) = w^*_L(x, 1+d) = 0, \quad x \in [0,1]. \end{cases}$$

The boundary function $(u-v)(0,y)$ is extended so that $(u-v)^*(0,y) = 0$ for $y < -\frac{d}{2}$ and $y > 1 + \frac{d}{2}$. By the standard comparison principle, it follows that

$$|w_L^*(x,y)| \le Ce^{-\sqrt{\frac{\gamma\alpha}{\varepsilon}}x}, \quad (x,y) \in \overline{\Omega}^{**}.$$

Note that the crude derivative bounds given in Lemma 1 also apply in the case when $a_1(x,y) \ge 0,\ a_2(x,y) \ge 0$ and hence they are applicable in the case of w_L^*, if the extensions are such that $a_1^* \ge 0,\ a_2^* \ge 0,\ b^* > 0$. In the direction orthogonal to the layer we sharpen these bounds. We first obtain a bound on w_L^* to reflect the fact that it is zero on the edges Γ_T^{**} and Γ_B^{**}. The coefficient a_2 is extended to the domain Ω^{**} so that

$$|a_2^*|_{\Omega^{**}} \le C_1\|a_2\|_{\Omega}(d+y)(1+d-y).$$

Assuming that μ is sufficiently small, we get that

$$|w_L^*(x,y)| \le C(d+y)(1+d-y), \quad (x,y) \in \overline{\Omega}^{**}. \tag{3.1}$$

From the above bound on $|w_L^*(x,y)|$ and the fact that $w_L^*(x,-d) = w_L^*(x,1+d) = 0$, we obtain

$$\left|\frac{\partial w_L^*}{\partial y}(0,y)\right| \le C, \qquad \frac{\partial w_L^*}{\partial y}(1,y) = 0,$$

$$\left|\frac{\partial w_L^*}{\partial y}(x,-d)\right| \le C, \quad \left|\frac{\partial w_L^*}{\partial y}(x,1+d)\right| \le C.$$

Differentiate the equation $L_{\varepsilon,\mu}^{**} w_L^* = 0$ with respect to y to obtain

$$L_{\varepsilon,\mu}^{**}\frac{\partial w_L^*}{\partial y} = -\mu\frac{\partial a_1^*}{\partial y}\frac{\partial w_L^*}{\partial x} - \mu\frac{\partial a_2^*}{\partial y}\frac{\partial w_L^*}{\partial y} + \frac{\partial b^*}{\partial y}w_L^* = \tilde{f}.$$

Using these bounds on the extended domain and $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$, we have $||\tilde{f}|| \le C$ and therefore

$$\left\|\frac{\partial w_L^*}{\partial y}\right\| \le C.$$

This argument can be extended to produce the higher derivative bounds

$$\left\|\frac{\partial^i w_L^*}{\partial y^i}\right\| \le C(1+\sqrt{\varepsilon}^{\ 1-i}), \qquad i = 2,3.$$

Define the boundary layer function w_L by

$$L_{\varepsilon,\mu} w_L = 0,\ (x,y) \in \Omega, \quad w_L|_{\Gamma_L} = u - v,\ w_L|_{\Gamma_R} = 0,\ w_L|_{\Gamma_T \cup \Gamma_B} = w_L^*.$$

Define the boundary layer functions associated with the other three edges w_T, w_R and w_B analogously.

Associated with the corner $\Gamma_{LB} = \Gamma_L \cap \Gamma_B$ define a corner layer function w_{LB} such that

$$\begin{cases} L_{\varepsilon,\mu} w_{LB} = 0\ (x,y) \in \Omega, \\ w_{LB} = -w_B,\ (x,y) \in \Gamma_L, \quad w_{LB} = -w_L,\ (x,y) \in \Gamma_B, \\ w_{LB} = 0,\ (x,y) \in \Gamma_R, \quad w_{LB} = 0,\ (x,y) \in \Gamma_T. \end{cases}$$

Note at the corner $(0,0)$, $w_L(x,0)$ is compatible with $w_L(0,y) = (u-v)(0,y)$ which is compatible with $(u-v)(x,0) = w_B(x,0)$ which in turn is compatible with $w_B(0,y)$. Hence $w_L(x,0)$ is compatible with $w_B(0,y)$ at $(0,0)$.

By using the comparison principle and the obvious barrier function, the following bound on w_{LB} holds

$$|w_{LB}(x,y)| \le Ce^{-\sqrt{\frac{\gamma\alpha}{\epsilon}}x}e^{-\sqrt{\frac{\gamma\alpha}{\epsilon}}y}.$$

Analogous bounds hold for the three other corners. In summary we state the main result of this paper:

Theorem 1. *When $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$, the solution u of* (1.1) *can be decomposed into the following sum of components*

$$u = v + w_L + w_R + w_T + w_B + w_{LB} + w_{LT} + w_{RB} + w_{RT},$$

where $L_{\varepsilon,\mu}v = f$, and the layer and corner layer functions are each solutions of the homogeneous equation $L_{\varepsilon,\mu}w = 0$. Boundary conditions for these functions can be specified so that the bounds on the components and their derivatives given below hold:

$$\left\|\frac{\partial^{k+m}v}{\partial x^k \partial y^m}\right\| \le C\big(1+\varepsilon^{\frac{2-k-m}{2}}\big), \quad 0 \le k+m \le 3,$$

$$|w_L(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x} \quad , \quad |w_B(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}y},$$

$$|w_R(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} \quad , \quad |w_T(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)},$$

$$\left\|\frac{\partial^k w_L}{\partial y^k}\right\| \le C\left(1+\sqrt{\varepsilon}^{\ 1-k}\right) \quad , \quad \left\|\frac{\partial^k w_R}{\partial y^k}\right\| \le C\left(1+\sqrt{\varepsilon}^{\ 1-k}\right),$$

$$\left\|\frac{\partial^k w_B}{\partial x^k}\right\| \le C\left(1+\sqrt{\varepsilon}^{\ 1-k}\right) \quad , \quad \left\|\frac{\partial^k w_T}{\partial x^k}\right\| \le C\left(1+\sqrt{\varepsilon}^{\ 1-k}\right),$$

$$|w_{LB}(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x}e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}y} \quad , \quad |w_{LT}(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x}e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)},$$

$$|w_{RB}(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)}e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}y} \quad , \quad |w_{RT}(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)}e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)}.$$

For all the layer components, we also have that

$$\left\|\frac{\partial^{k+m}w}{\partial x^k \partial y^m}\right\| \le C\varepsilon^{\frac{-k-m}{2}}, \ 1 \le k+m \le 3.$$

4. Numerical Method

Consider the following upwind finite difference scheme

$$L^{N,M}U(x_i,y_j) = \varepsilon\delta_x^2 U + \varepsilon\delta_y^2 U + \mu a_1 D_x^+ U + \mu a_2 D_y^+ U - bU = f,$$

where D^+ is the forward difference operator and δ^2 is the standard second order centered difference operator. We apply the above finite difference operator on the tensor product mesh $\Omega^{N,M} = \Omega^N \times \Omega^M$, where Ω^N (Ω^M) is a piecewise uniform mesh [1] that places a uniform mesh containing $\mathcal{O}(N)$ mesh points in each of the three subregions $[0, \sigma_x], [\sigma_x, 1-\sigma_x], [1-\sigma_x, 1]$. The transition points σ_x, σ_y are taken to be

$$\sigma_x = \min\Big\{\frac{1}{4},\ 2\sqrt{\frac{\varepsilon}{\gamma\alpha}}\ln N\Big\}, \quad \sigma_y = \min\Big\{\frac{1}{4},\ 2\sqrt{\frac{\varepsilon}{\gamma\alpha}}\ln M\Big\}.$$

Note that from the pointwise bounds on the layer components and for this choice of transition point, when $\sigma_x < \frac{1}{4}$,

$$\|w_L(x_i, y_j)\| \le CN^{-2},\ x_i \ge \sigma_x.$$

The discrete solution is decomposed into the sum

$$U = V + W_L + W_R + W_T + W_B + W_{LB} + W_{LT} + W_{RB} + W_{RT}.$$

where

$$L^{N,M}V = f,\ V|_{\Gamma^{N,M}} = v|_{\Gamma^{N,M}}, \quad L^{N,M}W_L = 0,\ W_L|_{\Gamma^{N,M}} = w_L|_{\Gamma^{N,M}},$$

and the other layer functions are defined similarly.

The maximum pointwise error $\|u - U\|$ is estimated by bounding each of the error components $\|v - V\|, \|w_L - W_L\|, \|w_R - W_R\| \ldots$ separately. The error $\|v - V\|$ is bounded using a classical truncation error and comparison principle argument. When $\sigma = \frac{1}{4}$ this classical argument is also used to bound the error in the layer components. For the case when $\sigma < \frac{1}{4}$, we have the following bounds on the discrete boundary layer function W_L

$$|W_L(x_i, y_j)| \le C\prod_{s=1}^{i}\left(1 + \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}h_s\right)^{-1}, \quad h_s = x_s - x_{s-1}$$

Outside the associated layer region and when $\sigma_x < \frac{1}{4}$, this bound is used to show that

$$\|(w_L - W_L)(x_i, y_j)\| \le CN^{-1}, x_i \ge \sigma_x.$$

Note that, for $x_i < \sigma_x$, the truncation error is

$$|L^{N,M}(w_L - W_L)| \le CN^{-1}\ln N\sqrt{\varepsilon}\left(\varepsilon\left\|\frac{\partial^3 w_L}{\partial x^3}\right\| + \mu\left\|\frac{\partial^2 w_L}{\partial x^2}\right\|\right)$$
$$+CM^{-1}\left(\varepsilon\left\|\frac{\partial^3 w_L}{\partial y^3}\right\| + \mu\left\|\frac{\partial^2 w_L}{\partial y^2}\right\|\right)$$

which implies that

$$\|w_L - W_L\| \le CN^{-1}\ln N + CM^{-1}.$$

The error in the other layer components are bounded in an analogous fashion.

Lemma 2. *Let u be the solution of the differential equation* (1.1) *and U be the discrete solution defined above. Then at each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$*

$$|(U - u)(x_i, y_j)| \le CN^{-1}\ln N + CM^{-1}\ln M.$$

where C is a constant independent of ε, μ and N.

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