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MULTIPLE SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS, WHICH HAVE OSCILLATORY SOLUTIONS

S. OGORODNIKOVA and F. SADYRBAEV

Daugavpils University

Parades str.1, Daugavpils LV-5400, Latvia

E-mail: oglana@tvnet.lv; felix@dau.lv

Abstract. First we consider two second order autonomous differential equations with critical points, which allow to detect an exact number of solutions of the Dirichlet boundary value problem. Then non-autonomous equations with similar behavior of solutions are considered. Estimations from below of the number of solutions to the Dirichlet boundary value problem are given.

Key words: singular points, multiple solutions, heteroclinic solutions, homoclinic solutions

1. Introduction

In the work [1, Ch. 15] estimations of the number of solutions to the boundary value problem

$$x' = h(t, x, y), \quad y' = f(t, x, y), \tag{1.1}$$

$$a_1 x(a) - b_1 x'(a) = 0, \tag{1.2}$$
$$a_2 x(b) - b_2 x'(b) = 0$$

were obtained. These estimations were based on comparison of the behavior of solutions in some neighborhood of the zero solution and at infinity. Notice that the zero solution exists since $h(t, 0, 0) = f(t, 0, 0) = 0$. It is convenient to explain the result of Perov in terms of the angular function $\varphi(t)$, which can be introduced by the relations

$$x = \rho \sin \varphi, \quad y = \rho \cos \varphi, \quad \rho^2 = x^2 + y^2. \tag{1.3}$$

One gets the following equations for the functions φ and ρ :

$$\begin{cases} \varphi' = \dfrac{1}{\rho}\,[h\,\cos\varphi - f\,\sin\varphi], \\ \rho' = \quad h\,\sin\varphi + f\,\cos\varphi. \end{cases} \tag{1.4}$$

Let φ_0 and φ_1 be the angles which relate respectively to the first and the second boundary conditions (1.2). Let us set

$$\rho_0 = \sqrt{x^2(a) + y^2(a)}.$$

Suppose that a solution $\varphi(t)$ of the system (1.4), which is defined by the initial condition $\varphi(a) = \varphi_0$ for $\rho_0 \sim 0$, takes exactly m values of the form $\varphi_1(\operatorname{mod}\pi)$. Moreover, assume that a solution $\varphi(t)$, which is defined by the initial condition $\varphi(a) = \varphi_0$ and which relates to values $\rho_0 \sim +\infty$, takes n values of the form $\varphi_1(\operatorname{mod}\pi)$. Then there exist at least $2|n-m|$ nontrivial solutions of the problem.

Fig.1 visualizes the case of $n = 0$ and $m = 1$. Two possible solutions of the BVP are represented by two semicircles.

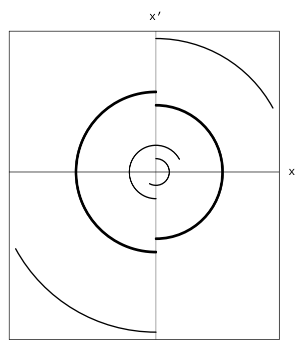

Figure 1. Perov's result ($m = 1,\ n = 0$), bold lines denote orbits of solutions of BVP; normal lines denote orbits at infinity and at zero.

Due to different rates of whirling of solutions near the zero and at infinity multiple solutions of the problem appear.

The above mentioned result by A. Perov is much more general than that described by Fig.1, since equations in (1.1) are non-autonomous.

Our aim in this paper is the following. We consider the second order equations, which are equivalent to two-dimensional systems, which are similar to those treated by A. Perov and which, moreover, can have hetero- and homoclinic type solutions. First, we consider autonomous equations which have singular points of the type saddle-center-saddle. Such equation has a heteroclinic solution and it may have multiple solutions of the Dirichlet problem. The obtained results are then generalized to the case of non-autonomous equation, which has a solution, defined on a finite interval and which possesses some properties of a heteroclinic solution.

Similar situation is considered for autonomous equations which have singular points of the type focus-saddle. This equation has a homoclinic solution and it may also have multiple solutions of the Dirichlet problem.

2. Autonomous Equations I

Consider the equations

$$\begin{aligned} x'' &= -x + x^3, \\ x'' &= -x + x^2. \end{aligned} \tag{2.1}$$

We will show that the Dirichlet boundary value problems for equations (2.1) have different numbers of solutions.

Consider the problem

$$\begin{cases} x'' = -\alpha x + x^3, \\ x(0) = 0, \quad x(1) = 0, \end{cases} \tag{2.2}$$

where the parameter α is positive. The equivalent system

$$\begin{cases} x' = \quad\quad y, \\ y' = -\alpha x + x^3 \end{cases} \tag{2.3}$$

has a center at $(0;0)$ and two saddle points at $(-\sqrt{\alpha};0)$ and $(\sqrt{\alpha};0)$. The heteroclinic orbit connects two saddle points. The respective heteroclinic solution has "an infinite" period [2]. The phase portrait of the solution is presented in Fig. 2.

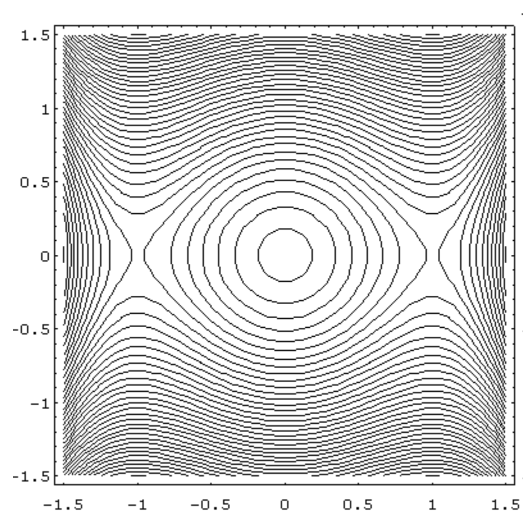

Figure 2. Visualization of Phase portrait.

Proposition 1. *Let the condition*

$$\pi^2 n^2 < \alpha < \pi^2 (n+1)^2 \tag{2.4}$$

hold, where n is a non-negative integer. Then problem (2.2) *has exactly* $2n$ *nontrivial solutions.*

3. Non-Autonomous Equations I

Consider the BV problem

$$\begin{cases} x'' = f(t,x), \\ x(0) = x(1) = 0, \end{cases} \tag{3.1}$$

where function f satisfies the following conditions:

(A1) f and f_x are $C(I \times R)$-functions;

(A2) $f(t,0) \equiv 0$;

(A3) $xf(t,x) > 0$ for $t \in I, |x| > M$, where $M > 0$ is constant;

(A4) there exists a solution $\eta(t)$ of the problem (3.1), $\eta(0) = 0, \eta'(0) > 0$ such that $\eta(t)$ does not vanish in the interval $(0;1]$;

(A5) there exists a solution $\xi(t)$ of the problem (3.1), $\xi(0) = 0, \xi'(0) < 0$ such that $\xi(t)$ does not vanish in the interval $(0;1]$;

(A6) solutions of equation (3.1) extend to the interval $(0;1]$.

Theorem 1. *Let the conditions **(A1)** – **(A6)** hold. Assume also that solutions $y(t)$ of the Cauchy problem*

$$\begin{cases} y'' = f_x(t,0)y, \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

have exactly n zeros in the interval $(0,1)$ and $y(1) \neq 0$. Then problem (3.1) *has at least* $2n$ *nontrivial solutions.*

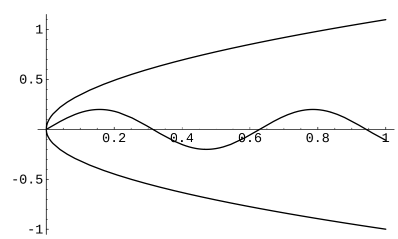

Figure 3. Visualization of Theorem 1.

4. Autonomous Equations, II

Consider the problem

$$\begin{cases} x'' = -\alpha x + x^2, \\ x(0) = 0, \quad x'(0) = 1, \end{cases} \tag{4.1}$$

where the parameter α is positive. The equivalent system

$$\begin{cases} x' = \quad\quad y, \\ y' = -\alpha x + x^2 \end{cases} \tag{4.2}$$

has a focus at $(0; 0)$ and a saddle point at $(\alpha; 0)$. The homoclinic orbit connects the saddle point to itself. It has “an infinite” period. A phase portrait is presented in Fig. 4.

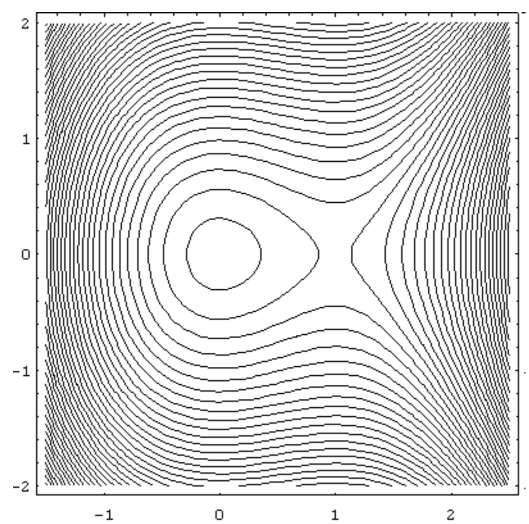

Figure 4. Visualization of Phase portrait.

Proposition 2. *Suppose that the condition*

$$\pi^2 n^2 < \alpha < \pi^2 (n+1)^2 \tag{4.3}$$

hold, where n is a positive integer. Then problem (4.1) *has exactly* $2n - 1$ *nontrivial solutions.*

5. Non-Autonomous Equations II

Let us consider the problem

$$\begin{cases} x'' = f(t, x), \\ x(0) = x(1) = 0, \end{cases} \tag{5.1}$$

where function f satisfies the conditions:

(A1) $f, f_x \in C(I \times R)$;

(A2) $f(t, 0) \equiv 0$;

(B) $f(t, x) > c|x|^p$ for $t \in I$, $|x| > M$, where $c > 0,\ p > 1,\ M > 0$ are constants;

(A4) there exists a solution $\eta(t)$ of the problem (5.1), $\eta(0) = 0,\ \eta'(0) > 0$ such that $\eta(t)$ does not vanish in the interval $(0; 1]$;

(A6) solutions of equation (3.1) extend to the interval $(0; 1]$.

Theorem 2. *Let the conditions **(A1)**, **(A2)**, **(B)**, **(A4)** and **(A6)** hold. Suppose that a solution $y(t)$ of the Cauchy problem*

$$\begin{cases} y'' = f_x(t, 0)y, \\ y(0) = 0, \quad y'(0) = 1 \end{cases} \tag{5.2}$$

has exactly $n \geq 1$ zeros in the interval $(0, 1)$ and $y(1) \neq 0$. Then the problem (5.1) has at least $2n - 1$ solutions.

The statement of this theorem is illustrated in Fig. 5.

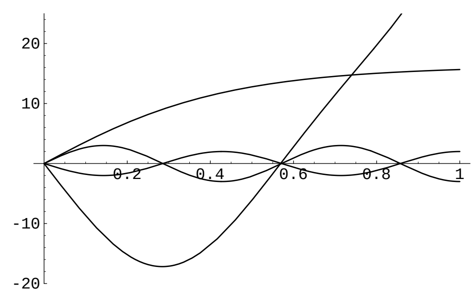

Figure 5. Visualization of Theorem 2.

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