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CHARACTERISTIC NUMBERS OF NON-AUTONOMOUS EMDEN-FOWLER TYPE EQUATIONS

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Abstract. Consider the Emden – Fowler equation $x'' = -q(t)|x|^{2\varepsilon}x, \quad \varepsilon > 0$, in the interval $[a, b]$. The coefficient $q(t)$ is a positive valued continuous function. The Nehari's characteristic number λ_n associated with the Emden – Fowler equation coincides with a minimal value of the functional $\frac{\varepsilon}{1+\varepsilon}\int_a^b x'^2(t)\,dt$ over all solutions of the boundary value problem

$$x'' = -q(t)|x|^{2\varepsilon}x, \quad x(a) = x(b) = 0, \quad x(t) \text{ has exactly } n-1 \text{ zeros in } (a,b).$$

The respective solution is called by Nehari's solution. We construct an example which shows that the Nehari's extremal problem may have more than a unique solution.

Key words: Characteristic numbers, Emden - Fowler equation, Nehari's solutions

1. Nehari's Solutions

Behavior of solutions to the Emden – Fowler type equation

$$x'' = -q(t)|x|^{2\varepsilon}x, \quad \varepsilon > 0, \tag{1.1}$$

where $q(t)$ is a positive valued continuous function, may be complicated if $q(t)$ is a non-monotone function.

Some regularity to the theory of the Emden – Fowler type equations of the form (1.1) is brought by the so called Nehari's solutions.

The Nehari's theory applies to equations of the type (1.1).

The general theorem by Nehari ([2, Theorem 3.2]) when adapted to the case under consideration states that the extremal problem below has a solution.

Problem:

$$H(x) = \int_a^b \left[x'^2 - (1+\varepsilon)^{-1} q(t) x^{2+2\varepsilon}\right] dt \to \inf, \quad x \in \Gamma_n, \tag{1.2}$$

where Γ_n consists of all functions $x(t)$, which are continuous and piece-wise continuously differentiable in $[a, b]$; there exist numbers a_ν such that

$$a = a_0 < a_1 < \ldots < a_n = b;$$

$x(a_0) = 0$ and for $\nu = 1, \ldots, n$, $x(a_\nu) = 0$ but $x \not\equiv 0$ in any $[a_{\nu-1}, a_\nu]$, and

$$\int_{a_{\nu-1}}^{a_\nu} x'^2(t)\, dt = \int_{a_{\nu-1}}^{a_\nu} q(t) x^2 |x|^{2\varepsilon}\, dt. \tag{1.3}$$

The respective extremal functions $x_n(t)$ are those solutions of equation (1.1), which vanish at the points $t = a$ and $t = b$, have exactly $n - 1$ zeros in (a, b) and satisfy the condition

$$\int_a^b x'^2\, dt = \int_a^b q(t) x^2 |x|^{2\varepsilon}\, dt. \tag{1.4}$$

By combining (1.3) with (1.4) one gets

$$\lambda_n(a, b) := \min_{x \in \Gamma_n} H(x) = H(x_n) = \frac{\varepsilon}{1+\varepsilon} \int_a^b q(t) x^{2+2\varepsilon}\, dt = \frac{\varepsilon}{1+\varepsilon} \int_a^b x'^2(t)\, dt.$$

Thus the characteristic number $\lambda_n(a, b)$ is (up to a multiplicative constant) a minimal value of the functional $\int_a^b x'^2(t)\, dt$ over the set of all solutions of the boundary value problem

$$x'' = -q(t)|x|^{2\varepsilon} x, \quad x(a) = x(b) = 0, \quad x(t) \text{ has } n-1 \text{ zeros in } (a, b).$$

We will call the characteristic numbers λ_n by *the Nehari's numbers* and the respective solutions of the differential equation by *the Nehari's solutions.*

Remark 1. Nehari's numbers $\lambda_n(a, b)$ are uniquely defined by the interval (a, b). In the work [2] Nehari mentioned that the theory could be developed mush easier if the associated Nehari's solution be unique. It was shown theoretically in [3] that this is not the case. There exist equations of the type (1.1), which have more than one Nehari's solution for certain a, b and n.

2. Example: Nonuniqueness of the Nehari's Solutions

We construct the Emden - Fowler equation which possesses two Nehari's solutions.

In our considerations we use systematically the lemniscatic functions $\operatorname{sl} t$ and $\operatorname{cl} t$ which can be defined as solutions of the equation $x'' = -2x^3$, subject to the initial conditions $x(0) = 0, \quad x'(0) = 1$ and $x(0) = 1, \quad x'(0) = 0$ respectively. Both functions are periodic with a minimal period of $4A$, where $A = \int_0^1 \frac{ds}{\sqrt{1-s^4}}$. One may consult the paper [1] for more properties of these functions. In many respects they behave like usual trigonometric functions.

Equation. Consider equation

$$x'' = -q(t)\, x^3, \quad t \in (-1, 1), \tag{2.1}$$

together with the boundary conditions

$$x(-1) = 0, \quad x(1) = 0, \quad x(t) > 0,\ t \in (-1, 1). \tag{2.2}$$

The coefficient $q(t)$ is constructed as follows. Let

$$q(t) = \frac{2}{\big(\xi(t)\big)^6},$$

where

$$\xi(t) = \begin{cases} \xi_1(t),\ -1 \le t \le 0, \\ \xi_2(t),\ 0 \le t \le 1 \end{cases}$$

and

$$\xi_1(t) = ht + \eta, \quad -1 \le t \le 0,$$
$$\xi_2(t) = -ht + \eta, \quad 0 \le t \le 1.$$

Thus $\xi(t)$ is a "Λ-shaped" piece-wise linear function, which depends on a positive valued parameter h, η depends on h as $\eta = h + 1$.

Solutions. Our goal: we are looking for a solution (solutions) of the problem (2.1), (2.2).

Consider two problems

$$\begin{aligned} &x_1'' = -\frac{k}{(ht+\eta)^6}\, x_1^3, \quad x_1(-1) = 0,\ x_1(0) = \tau, \quad x_1(t) > 0,\ t \in (-1, 0); \\ &x_2'' = -\frac{k}{(-ht+\eta)^6}\, x_2^3, \quad x_2(0) = \tau,\ x_2(1) = 0, x_2(t) > 0,\ t \in (0, 1), \end{aligned} \tag{2.3}$$

where $\tau > 0$. Let

$x_1(t)$ be a solution of the first equation of (2.3) in $[-1; 0]$;
$x_2(t)$ be a solution of the second equation of (2.3) in $[0; 1]$.

Then the function

$$x(t) = \begin{cases} x_1(t), \text{ if } -1 \le t \le 0, \\ x_2(t), \text{ if } 0 \le t \le 1 \end{cases}$$

is a C^2-solution of the problem (2.1), (2.2) if additionally the smoothness condition

$$x_1'(0) = x_2'(0)$$

is satisfied. The problem (2.3) can be explicitly resolved as

$$x_1(t, \beta_1) = \beta_1^{\frac{1}{2}}(ht + \eta) \cdot \mathrm{sl}\left(\beta_1^{\frac{1}{2}} \frac{t+1}{ht+\eta}\right),$$

where

$$\beta_1 = x_1'(-1) > 0$$

and

$$x_1(0; \beta_1) = \tau.$$

The derivative is given by

$$x_1'(t; \beta_1) = \beta_1^{\frac{1}{2}} h \cdot \mathrm{sl}\left(\beta_1^{\frac{1}{2}} \frac{t+1}{ht+\eta}\right) + \beta_1 \frac{-h+\eta}{ht+\eta} \cdot \mathrm{sl}'\left(\beta_1^{\frac{1}{2}} \frac{t+1}{ht+\eta}\right).$$

Similar formulas are valid for $x_2(t)$. Notice that $x_2'(1) = -\beta_2 < 0$. In order to get an explicit formula for a solution of the BVP (2.1), (2.2) one have to solve a system of two equations with respect to (β_1, β_2)

$$x_1(0; \beta_1) = x_2(0; \beta_2),$$

$$x_1'(0; \beta_1) = x_2'(0; \beta_2).$$

This system after replacements and simplifications looks as

$$\begin{cases} \beta_1^{\frac{1}{2}} \cdot \mathrm{sl}\left(\dfrac{\beta_1^{\frac{1}{2}}}{\eta}\right) = \beta_2^{\frac{1}{2}} \cdot \mathrm{sl}\left(\dfrac{\beta_2^{\frac{1}{2}}}{\eta}\right), \\ \beta_1^{\frac{1}{2}} h \cdot \mathrm{sl}\left(\dfrac{\beta_1^{\frac{1}{2}}}{\eta}\right) + \dfrac{\beta_1}{\eta} \cdot \mathrm{sl}'\left(\dfrac{\beta_1^{\frac{1}{2}}}{\eta}\right) = -\beta_2^{\frac{1}{2}} h \cdot \mathrm{sl}\left(\dfrac{\beta_2^{\frac{1}{2}}}{\eta}\right) - \dfrac{\beta_2}{\eta} \cdot \mathrm{sl}'\left(\dfrac{\beta_2^{\frac{1}{2}}}{\eta}\right), \end{cases}$$

where $0 < \frac{\beta_1^{\frac{1}{2}}}{\eta}, \frac{\beta_2^{\frac{1}{2}}}{\eta} < 2A$. In new variables $u := \frac{\beta_1^{\frac{1}{2}}}{\eta}, \quad v := \frac{\beta_2^{\frac{1}{2}}}{\eta}$ the system takes the form

$$\begin{cases} u \, \mathrm{sl}\, u = v \, \mathrm{sl}\, v, \quad 0 < u, v < 2A, \\ hu \, \mathrm{sl}\, u + u^2 \, \mathrm{sl}'\, u = -hv \, \mathrm{sl}\, v - v^2 \, \mathrm{sl}'\, v, \quad h > 0. \end{cases} \tag{2.4}$$

Notice that if a solution $(\bar{u}, \bar{v})$ of the system (2.4) exists, then a solution $x(t)$ of the BVP (2.1), (2.2) can be constructed such that

$$x'(-1) = \beta_1 = \bar{u}^2(h+1)^2, \quad x'(1) = -\beta_2 = -\bar{v}^2(h+1)^2.$$

Proposition 1. *For h large the system* (2.4) *has exactly three solutions, which have the following characteristics.*

1. *There exists a unique symmetric solution* (u_0, v_0), *that is,* $u_0 = v_0$. *One has that* $(u_0, v_0) \to (2A, 2A)$ *as* $h \to +\infty$.
2. *There exists a unique solution* (u_1, v_1) *in the triangle* $\{0 < u, v < 2A,\ v > u)\}$ *for h large. Moreover,* $(u_1, v_1) \to (0, 2A)$ *as* $h \to +\infty$.
3. *There exists a unique solution* (u_2, v_2) *in the triangle* $\{0 < u, v < 2A,\ v < u)\}$ *for h large. Solutions* (u_1, v_1) *and* (u_2, v_2) *are symmetric, that is,* $(v_2, u_2) = (u_1, v_1)$.

Zeros of the functions $\Phi(u, v) = u \operatorname{sl} u - v \operatorname{sl} v$ and $\Psi(u, v) = hu \operatorname{sl} u + u^2 \operatorname{sl}' u + hv \operatorname{sl} v + v^2 \operatorname{sl}' v$ in the square $Q = \{(u, v) :\ 0 \le u, v \le 2A\}$ for $h > 1$ are depicted in the Figure 1. Notice that a set of zeros of Φ consists of the diagonal $u = v$ and two symmetric branches.

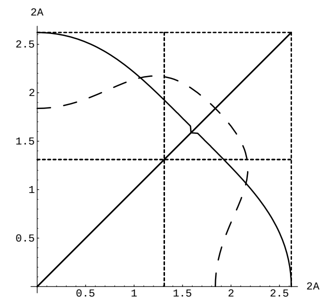
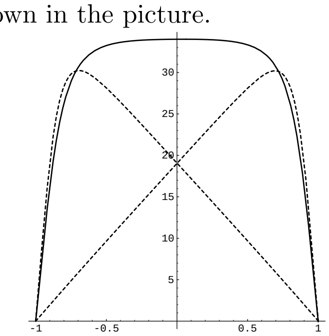


Figure 1. Zeros of $\Phi(u, v)$ (solid line) and $\Psi(u, v)$ (dashed line), $h = 2$.

Nehari's numbers. The respective solutions of the boundary value problem (2.1), (2.2) look as shown in the picture.



Let $H = \frac{1}{2} \int_{-1}^{1} x'^2(t)dt$. Denote by H_{sym} and H_{asym} the respective values of H for a symmetric solution (which is depicted by solid line), and for asymmetric solutions (depicted by dashed lines). Notice that $H(x)$ is the same for both asymmetric solutions. The values of H_{sym} and H_{asym} are

$$
\begin{aligned}
H_{sym} &= \tfrac{2}{3}u_0^{\frac{3}{2}}\left(\tfrac{u_0^{\frac{1}{2}}}{h+1} - \mathrm{sl}'\left(\tfrac{u_0^{\frac{1}{2}}}{h+1}\right)\mathrm{sl}\left(\tfrac{u_0^{\frac{1}{2}}}{h+1}\right)\right), \\
H_{asym} &= \tfrac{1}{3}u_1^{\frac{3}{2}}\left(\tfrac{u_1^{\frac{1}{2}}}{h+1} - \mathrm{sl}'\left(\tfrac{u_1^{\frac{1}{2}}}{h+1}\right)\mathrm{sl}\left(\tfrac{u_1^{\frac{1}{2}}}{h+1}\right)\right) \\
&\quad + \tfrac{1}{3}v_1^{\frac{3}{2}}\left(\tfrac{v_1^{\frac{1}{2}}}{h+1} - \mathrm{sl}'\left(\tfrac{v_1^{\frac{1}{2}}}{h+1}\right)\mathrm{sl}\left(\tfrac{v_1^{\frac{1}{2}}}{h+1}\right)\right).
\end{aligned}
$$

Proposition 2.

$$\frac{H_{sym}}{H_{asym}} \xrightarrow[h\to+\infty]{} 2.$$

Therefore for h large two asymmetric solutions are the Nehari's solutions.

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