Mathematical Modelling and Analysis 2005. Pages 519–524
Proceedings of the 10^{th} International Conference MMA2005&CMAM2, Trakai
 ISBN 9986-05-924-0

THE R-FUNCTIONS METHOD FOR SOLUTION OF MULTICONNECTED STOKES FLOWS IN DOMAINS OF COMPLICATED GEOMETRY

YU. SEMERICH[1] and M. SIDOROV[2]

[1] *A.N. Podgorniy's Institute for Problems in Mechanical Engineering of the National Academy of Sciences of Ukraine*

Dm. Pozhgarskogo str., 2/10, 61046, Kharkov, Ukraine

E-mail: `semerich@ipmach.kharkov.ua`

[2] *Kharkov National University of Radioelectronics*

Lenin av., 14, 61166, Kharkov, Ukraine

E-mail: `maxim@kture.kharkov.ua`

Abstract. This paper deals with the investigation of stationary flow for a viscous incompressible liquid in multiconnected domains. The Stokes approach and boundary value problem for a flow function are considered. The R-functions and Ritz methods are used to solve the boundary value problem.

Key words: Stokes approach, R-functions method, general structure of solution

1. Statement of the Problem

Consider a stationary flow of a viscous incompressible liquid in a $(m+1)$-connected plane domain Ω with an external boundary $\partial\Omega_0$ and internals ones $\partial\Omega_1, \partial\Omega_2, ..., \partial\Omega_n$. Suppose $\partial\Omega = \bigcup\limits_{i=0}^{n} \partial\Omega_i$ and $\partial\Omega_k \cap \partial\Omega_j = \emptyset$, $k \neq j$, $k, j = 1, 2, ..., n$. The flow is described by the Stokes and continuity equations:

$$\begin{cases} -\nu \Delta \mathbf{v} = -\dfrac{1}{\rho}\nabla p + \mathbf{f}, \\ \operatorname{div} \mathbf{v} = 0, \quad \mathbf{v}|_{\partial\Omega} = \mathbf{v}^* . \end{cases}$$

Here $\mathbf{v}$ denotes the field of velocities, p is the pressure, $\mathbf{f}$ is the field of volume forces, ν is the kinematic viscosity, ρ is the density.

The field of velocities in the domain Ω are described by the equation for a flow function $\psi\,(x, y)$

$$\Delta^2\psi = \mathbf{Re}\left(\frac{\partial f_x}{\partial y} - \frac{\partial f_y}{\partial x}\right). \tag{1.1}$$

The flow function $\psi(x,y)$ is connected to the field components v_x and v_y by the expressions $v_x = \frac{\partial\psi}{\partial y}$, $v_y = -\frac{\partial\psi}{\partial x}$.

Suppose that a flow of mass over domains $\partial\Omega_k$, $k = 1,2,...,n$ is absent [2]:

$$\int\limits_{\partial\Omega_k} d\psi = 0, \quad k = 1,2,...,n.$$

It means that function $\psi(x,y)$ is one-valued.

The flow function $\psi(x,y)$ is defined only to within additive constant. Let fix value of the constant in some point $s_0 \in \partial\Omega_0$, i.e. suppose $\psi(s_0) = 0$. Let choose a function $\tilde{f}_0(s) \in W_2^1(\partial\Omega)$ such, that $\left.\frac{\partial\tilde{f}_0}{\partial\tau}\right|_{\partial\Omega} = v^*\cdot n$, $\tilde{f}(s_0) = 0$.

Since $\left.\dfrac{\partial\psi}{\partial\tau}\right|_{\partial\Omega} = v^*\cdot n$, then $\psi(x,y)$ coincides with $\tilde{f}_0(s)$ in $\partial\Omega_0$ and differs from it by some constant in $\partial\Omega_i$, $i = 1,2,...,n$. Thus,

$$\psi|_{\partial\Omega_0} = \tilde{f}_0(s), \tag{1.2}$$

$$\psi|_{\partial\Omega_i} = \tilde{f}_0(s) + \alpha_i, \quad i = 1,2,...,n, \ \ s\in\partial\Omega, \tag{1.3}$$

where α_i, $i = 1,2,...,n$ are fixed and unknown numbers.

Moreover,

$$\left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = \tilde{g}_0(s), s\in\partial\Omega, \tag{1.4}$$

where $\tilde{g}_0(s) = -v^*\cdot\tau$. It is proved in [1] that problem (1.1), (1.2) – (1.4) is solvable in $W_2^2(\Omega)$ at $f \in [L_2(\Omega)]^2$, $v^* \in [L_2(\partial\Omega)]^2$, $\int\limits_{\partial\Omega_i} v^*\cdot n ds = 0$, $i = 0,1,...,n$.

According to [2], the constants α_i can be determined as

$$\oint\limits_{\Gamma} \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0, \tag{1.5}$$

where Γ is the any closed contour, which belongs to $\overline{\Omega}$. In particular, (1.5) must be fulfilled for the contours $\partial\Omega_1, \partial\Omega_2, ..., \partial\Omega_n$. Since

$$\frac{\partial p}{\partial x} = f_x + \frac{1}{\mathbf{Re}}\Delta\left(\frac{\partial\psi}{\partial y}\right), \quad \frac{\partial p}{\partial y} = f_y - \frac{1}{\mathbf{Re}}\Delta\left(\frac{\partial\psi}{\partial x}\right)$$

then (1.5) can be written as

$$\oint\limits_{\Gamma}\left(f_x + \frac{1}{\mathbf{Re}}\Delta\left(\frac{\partial\psi}{\partial y}\right)\right)dx + \left(f_y - \frac{1}{\mathbf{Re}}\Delta\left(\frac{\partial\psi}{\partial x}\right)\right)dy = 0. \tag{1.6}$$

Taking into account the connection between line integrals of the first and second orders

$$\oint\limits_{L} a_x dx + a_y dy = \int\limits_{L} \mathbf{a} \cdot \tau ds, \qquad \tau = \left. \frac{\partial \omega_\Gamma}{\partial y} \mathbf{i} - \frac{\partial \omega_\Gamma}{\partial x} \mathbf{j} \right|_{\Gamma},$$

equation (1.6) can be written as

$$\int\limits_{\Gamma} \mathbf{f} \cdot \tau \, ds - \frac{1}{\mathbf{Re}} \int\limits_{\Gamma} \left[-\frac{\partial \omega_\Gamma}{\partial y} \frac{\partial \Delta \psi}{\partial y} - \frac{\partial \omega_\Gamma}{\partial x} \frac{\partial \Delta \psi}{\partial x} \right] ds$$
$$= \int\limits_{\Gamma} \mathbf{f}_\tau \, ds - \frac{1}{\mathbf{Re}} \int\limits_{\Gamma} \frac{\partial \Delta \psi}{\partial \mathbf{n}} ds = 0.$$

Here $\omega_\Gamma (x, y)$ such, that $\omega_\Gamma (x, y) = 0$, $(x, y) \in \Gamma$; $|\nabla \omega_\Gamma| = 1$, $(x, y) \in \Gamma$; $\mathbf{f}_\tau$, denotes the tangential component of the mass force vector $\mathbf{f}$, τ is the unit vector, $\mathbf{n}$ is the external to Γ normal vector. The vectors $\mathbf{n}$, τ, $\mathbf{n} \times \tau$ are right-oriented.

Thus, we have the problem to integrate equation (1.1) in domain Ω with the boundary conditions (1.2)–(1.4), constants α_i, $i = 1, 2, ..., n$ can be defined as

$$\int\limits_{\partial \Omega_i} \mathbf{f}_\tau ds = \frac{1}{\mathbf{Re}} \int\limits_{\partial \Omega_i} \frac{\partial \Delta \psi}{\partial \mathbf{n}} ds, i = 1, ..., n. \tag{1.7}$$

2. Method of the Problem Solving

Consider the R-functions method and superposition principle for solving the problem (1.1), (1.2)–(1.4), (1.7). Let write function $\psi (x, y)$ as

$$\psi (x, y) = \psi_0 (x, y) + \sum_{i=1}^{n} \alpha_i \psi_i (x, y), \tag{2.1}$$

where α_i, $i = 1, 2, ..., n$ are constants from (1.3), function $\psi_0 (x, y)$ is a solution of the following problem

$$\begin{cases} \Delta^2 \psi_0 = \mathbf{Re} \left(\dfrac{\partial f_x}{\partial y} - \dfrac{\partial f_y}{\partial x} \right), \quad (x, y) \in \Omega, \\ \psi_0|_{\partial \Omega} = \tilde{f}_0 (s), \quad \left. \dfrac{\partial \psi_0}{\partial n} \right|_{\partial \Omega} = \tilde{g}_0 (s), \quad s \in \partial \Omega. \end{cases} \tag{2.2}$$

Functions $\psi_i (x, y)$, $i = 0, 1, ..., n$ are solutions of the following problem

$$\begin{cases} \Delta^2 \psi_i = 0, \quad (x, y) \in \Omega, \\ \psi_i|_{\partial \Omega \setminus \partial \Omega_i} = 0, \quad \psi_i|_{\partial \Omega_i} = 1, \quad \left. \dfrac{\partial \psi_i}{\partial n} \right|_{\partial \Omega} = 0. \end{cases} \tag{2.3}$$

Obviously, selection of such functions $\psi_i(x,y)$, $i=0,1,...,n$ guarantees that function $\psi(x,y)$ of (2.1) satisfies equation (1.1) and the boundary conditions (1.2)–(1.4).

Let substitute (2.1) into expressions (1.7) for determining constants α_i, $i=1,2,...,n$, then we get the following system of linear algebraic equations

$$\frac{1}{\mathbf{Re}}\sum_{j=1}^{n}\alpha_j\int\limits_{\partial\Omega_i}\frac{\partial\Delta\psi_j}{\partial n}ds=-\frac{1}{\mathbf{Re}}\int\limits_{\partial\Omega_i}\frac{\partial\Delta\psi_0}{\partial n}ds+\int\limits_{\partial\Omega_i}f_\tau ds, i=1,2,...,n. \quad (2.4)$$

We will use the R-functions method for solving problems (2.2) and (2.3). Let define the functions $\omega_0(x,y)$, $\omega_1(x,y)$, ..., $\omega_n(x,y)$ such, that

1) $\omega_i(x,y)=0, \quad (x,y)\in\partial\Omega_i, \quad i=0,1,...,n,$

2) $\omega_i(x,y)>0, \quad (x,y)\in\Omega\cup\Big(\bigcup\limits_{\substack{j=0\\ j\neq i}}^{n}\partial\Omega_j\Big),$

3) $|\nabla\omega_i|=1, \quad (x,y)\in\partial\Omega_i, \quad i=0,1,...,n.$

According to the R-functions method, the structure of solution for the boundary value problem (2.2) has the form

$$\psi_0=f_0-\omega\left(D_1f_0+g_0\right)+\omega^2\Phi_0,$$

here $\omega=\bigwedge\limits_{i=1}^{n}{}_{\alpha}\,\omega_i$, $f_0=\mathrm{EC}\left(\tilde{f}_0\right)$, $g_0=\mathrm{EC}\left(\tilde{g}_0\right)$, $D_1f_0=(\nabla\omega,\nabla f_0)$, Φ_0 is the undefined component. Using the EC-operator [3], the boundary conditions of (2.3) can be written as

$$\psi_i|_{\partial\Omega}=\left.\frac{\frac{1}{\omega_i}}{\sum\limits_{j=0}^{n}\frac{1}{\omega_j}}\right|_{\partial\Omega}=f_i|_{\partial\Omega}, \quad \left.\frac{\partial\psi_i}{\partial n}\right|_{\partial\Omega}=0=g_i|_{\partial\Omega}.$$

The structure of solution for the boundary value problem (2.3) has the form

$$\psi_i=\frac{\frac{1}{\omega_i}}{\sum\limits_{j=0}^{n}\frac{1}{\omega_j}}-\omega D_1\left(\frac{\frac{1}{\omega_i}}{\sum\limits_{j=0}^{n}\frac{1}{\omega_j}}\right)+\omega^2\Phi_i=\varphi_i+\omega^2\Phi_i,$$

$$\varphi_i=\frac{\frac{1}{\omega_i}}{\sum\limits_{j=0}^{n}\frac{1}{\omega_j}}-\omega D_1\left(\frac{\frac{1}{\omega_i}}{\sum\limits_{j=0}^{n}\frac{1}{\omega_j}}\right),$$

where Φ_i, $i=1,2,...,n$ are undefined components.

Use the Ritz method to approximate the components Φ_i, $i=0,1,...,n$. Let define Φ_i, $i=0,1,...,n$ as $\Phi_i=\sum\limits_{k=1}^{N_i}c_{i,k}\phi_{i,k}$, where $\{\phi_{i,k}\}_{k=1}^{\infty}$ are complete systems of a functions, $i = 0, 1, ..., n$. Let systems of functions $\left\{\omega^2 \phi_{i,k}\right\}_{k=1}^{\infty}$ are coordinate sequences for any $i = 0, 1, ..., n$ such, that

1) $\omega^2 \phi_{i,k} \in \overset{\circ}{W}{}_2^2 (\Omega)\,, \quad k = 1, 2, ...,$

2) $\left\{\omega^2 \phi_{i,k}\right\}_{k=1}^{\infty}$ is complete in $\overset{\circ}{W}{}_2^2 (\Omega)$;

3) $\omega^2 \phi_{i,1}, \omega^2 \phi_{i,2}, ..., \omega^2 \phi_{i,N}$ are linearly independent for any N.

Let $\varphi_0 = f_0 - \omega (D_1 f_0 + g_0) \in W_2^2 (\Omega),\ F = \frac{\partial f_x}{\partial y} - \frac{\partial f_y}{\partial x} \in L_2 (\Omega)$, then problem (2.2) has only general solution $\psi_0 \in W_2^2 (\Omega)$ such, that $\psi_0 = \varphi_0 + u_0$, where function u_0 yields the minimum of the functional in $\overset{\circ}{W}{}_2^2 (\Omega)$

$$J_0 [u] = \iint\limits_{\Omega} \left[(\Delta u)^2 + 2\Delta u \Delta \varphi_0 - 2\mathbf{Re} u F\right] dxdy.$$

Let $\varphi_i = \frac{\frac{1}{\omega_i}}{\sum\limits_{j=0}^{n} \frac{1}{\omega_j}} - \omega D_1 \left(\frac{\frac{1}{\omega_i}}{\sum\limits_{j=0}^{n} \frac{1}{\omega_j}}\right) \in W_2^2 (\Omega)\,,\ i = 1, 2, ..., n$, then problem (2.3) has only general solution $\psi_i \in W_2^2 (\Omega)$ for all $i = 1, 2, ..., n$ such, that $\psi_i = \varphi_i + u_i$ where functions u_i yield the minimum of functionals in $\overset{\circ}{W}{}_2^2 (\Omega)$

$$J_i [u] = \iint\limits_{\Omega} \left[(\Delta u)^2 + 2\Delta u \Delta \varphi_i\right] dxdy.$$

Let approximate functions $u_i,\ i = 0, 1, ..., n$ as

$$u_i \approx u_{i,N_i} = \sum_{k=1}^{N_i} c_{i,k} \omega^2 \phi_{i,k}, i = 0, 1, ..., n,$$

where $c_{i,k},\ k = 1, 2, ..., N_i,\ i = 0, 1, ..., n$, are the undefined components. For their determination, according to the Ritz method, we have $(n + 1)$-systems of linear algebraic equations

$$\sum_{j=1}^{N_i} \left[\omega^2 \phi_{i,k}, \omega^2 \phi_{i,j}\right] c_{i,j} = - \left[\omega^2 \phi_{i,k}, \varphi_i\right], \tag{2.5}$$

where $[u, v] = \iint\limits_{\Omega} \Delta u \Delta v \, dxdy,\ k = 1, 2, ..., N_i,\ i = 0, 1, ..., n.$

Let ψ_i be such, that $\frac{\partial \Delta \psi_i}{\partial n} \in L_2 (\partial \Omega),\ i = 0, 1, ..., n$. Substituting the found values

$$\varphi_{i,N_i} = \varphi_i + u_{i,N_i} = \varphi_i + \sum_{k=1}^{N_i} c_{i,k} \omega^2 \phi_{i,k}$$

into (2.4), we get the system of linear algebraic equations for determination of the $\alpha_i,\ i = 1, 2, ..., n$.

Next we compute $\frac{\partial \Delta \psi_i}{\partial n}$. Since $\psi_j = \varphi_j + \omega^2 \Phi_j$, then

$$\Delta\psi_j = \Delta\varphi_j + \Delta\left(\omega^2\Phi_j\right) = \Delta\varphi_j + \omega^2\Delta\Phi_j + \Phi_j\left(2\omega\Delta\omega + |\nabla\omega|^2\right) + 2\omega\left(\nabla\omega, \nabla\Phi_j\right),$$

$$\frac{\partial\Delta\psi_j}{\partial n} = \frac{\partial\Delta\varphi_j}{\partial n} + 2\omega\frac{\partial\omega}{\partial n}\Phi_j + \omega^2\frac{\partial\Delta\Phi_j}{\partial n}\left(2\omega\Delta\omega + |\nabla\omega|^2\right) + \Phi_j\Big(2\frac{\partial\omega}{\partial n}\Delta\omega + 2\omega\frac{\partial\Delta\omega}{\partial n} + 2\Big(\nabla\omega, \nabla\frac{\partial\omega}{\partial n}\Big)\Big) + 2\frac{\partial\omega}{\partial n}\left(\nabla\omega, \nabla\Phi_j\right) + 2\omega\frac{\partial}{\partial n}\left(\nabla\omega, \nabla\Phi_j\right).$$

Taking into account $\omega|_{\partial\Omega} = 0$, $\frac{\partial\omega}{\partial n}\big|_{\partial\Omega} = -1$, $(\nabla\omega, \nabla f)|_{\partial\Omega} = \frac{\partial f}{\partial n}\Big|_{\partial\Omega}$, we have

$$\int\limits_{\partial\Omega_i}\frac{\partial\Delta\psi_j}{\partial n}\,ds = \int\limits_{\partial\Omega_i}\left(\frac{\partial\Delta\varphi_j}{\partial n} + 2\Phi_j\left(-\Delta\omega + \frac{\partial^2\omega}{\partial n^2}\right) - 2\frac{\partial\Phi_j}{\partial n}\right)ds$$

$$\approx \int\limits_{\partial\Omega_i}\left(\frac{\partial\Delta\varphi_j}{\partial n} + 2\left(-\Delta\omega + \frac{\partial^2\omega}{\partial n^2}\right)\sum_{k=1}^{N_j} c_{j,k}\phi_{j,k} - 2\sum_{k=1}^{N_j} c_{j,k}\frac{\partial\phi_{j,k}}{\partial n}\right)ds.$$

Thus, we have proved the following theorem.

Theorem 1. *If* $\varphi_i \in W_2^2(\Omega)$, $i = 0, 1, ..., n$, $F \in L_2\left(\overline{\Omega}\right)$, $\frac{\partial\Delta\psi_i}{\partial n} \in L_2(\partial\Omega)$, *where* ψ_i, $i = 0, 1, ..., n$ *are solutions of problems* (2.2), *then the sequence* $\psi_M = \psi_{0,N_0} + \sum\limits_{i=1}^{n}\alpha_{i,M}\psi_{i,N_i}$, $M = \sum\limits_{i=0}^{n} N_i$ *convergences to the only general solution of problem* (1.1), (1.2) – (1.4), (1.7) *at the* $\min\{N_0, N_1, ..., N_n\} \to \infty$ *in the norm of space* $W_2^2(\Omega)$.

The function ψ_M allows us to find approximation for the field of velocities and the pressure.

Numerical experiments were conducted for the rectangular domain with one and two obstacles. Mathematica was used for implementation of computations. It is supposed, that mass forces f are potential, i.e. $\mathrm{rot} f = 0$, it's mean $F(x, y) \equiv 0$. Cubic splines are used as basis functions for undefined components approximation.

References

[1] V. Girault and P.-A. Raviart. *Finite Element Approximation of the Navier-Stokes Equations.* Berlin, Heidelberg, New York: Springer-Verlag, 1979. 200

[2] C.E. Pearson. A computational methods for viscous flow problems. *J. Fluid Mech.*, **24**(4), 611 – 622, 1965.

[3] V.L. Rvachev and T.I.Sheiko. R-functions in boundary value problems in mechanics. *Appl. Mech. Rev.*, **48**(4), 151 – 188, 1995.