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DOMAIN DECOMPOSITION METHOD FOR SUBDOMAINS WITH NON-REGULAR MULTIPLE OVERLAPPING

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Abstract. The paper presents an outline of the proof of convergence of Schwarz Domain Decomposition algorithm for subdomains with non-regular multiple overlapping. This result is shown for linear PDE's fulfilling the maximum principle. The proof provides the sufficient condition, which describes in geometric terms the admissible structure of overlapping.

Key words: domain decomposition, Chimera method, overlapping domains

1. Introduction

Domain Decomposition is a method allowing to solve efficiently Partial Differential Equations. It is used for parallelization of numerical algorithms. Extensive literature exists dealing with the theory and numerical aspects of this method (e.g. [2, 3]). Domain decomposition, however, is used also to numerically solve PDE's in complex geometries, where generation of global grid may be very difficult (Chimera method). This is particularly true, in the presence of moving and free boundaries.

In this approach the complex computational domain is divided into the union of simpler subdomains, which may overlap in a non-regular manner. As a rule, multiple overlaps appear between groups of subdomains. The PDE is solved locally on each subdomain, while the global solution is obtained by iteratively adjusting the boundary conditions on each subdomain and repeating this procedure until the convergence is obtained.

Present method relies on using blending functions which allow the subdomains to overlap in almost arbitrary manner including multiple overlaps. This approach was first applied for numerical solution of Navier-Stokes and Euler equations in [1, 4]. However, to the authors knowledge, no proof of convergence of the Schwarz alternating algorithm for irregular overlapping was yet presented. The present paper introduces such proof, extending earlier ideas used for regular overlapping.

2. The General Problem

Let's consider the problem to find $u \in V(\Omega)$ such that

$$\begin{aligned} L(u) &= f, \\ u|_{\partial\Omega} &= g, \end{aligned} \tag{2.1}$$

where L is a linear operator, fulfilling the maximum principle

$$\inf(g) < R(\Omega, g, 0)(x) < \sup(g) \quad \forall x \in \Omega \setminus \partial\Omega, \tag{2.2}$$

where $R(\Omega, g, f)$ denotes the exact solution of the general problem (2.1).

3. Schwarz Method

We first choose a decomposition of the domain $\Omega = \cup\Omega_i$, $i = 1, \ldots, k$. To find the global solution $u^* = R(\Omega, g, f)$, the Schwarz method consists of the following steps.

- Set a first guess u_0 such that $u_0|_{\partial\Omega} = g$.
- Define a sequence for $n \geq 0$:

  $u^i_{n+1} = R(\Omega_i, u_n|_{\partial\Omega_i}, f)$,

  $u_{n+1} = \sum_{i=1}^{k} \chi_i u^i_{n+1}$.

The objective of the present paper is to prove convergence of this sequence to u^*. The blending functions χ_i has to fulfill conditions (see also [4]):

i) $0 \leq \chi_i \leq 1, \quad \sum_{i=1}^{k} \chi_i = 1 \quad i = 1, \ldots, k$,
ii) $\chi_i|_{\partial\Omega_i \setminus \partial\Omega} = 0$,
iii) $\mathrm{supp}(\chi_i) \subset \Omega_i$.

4. Some Observations

To proof the convergence, we need to set some geometrical condition to subdomains. They will result from the following observations:

1. The error on $\partial\Omega$ is zero in each step

$$(u^i_n - u^*)|_{\partial\Omega} = 0. \tag{4.1}$$

2. The error on the subdomain Ω_i in each step fulfill
$$u^i_{n+1} - u^* = R(\Omega_i, b, 0), \tag{4.2}$$
where $b = (u^i_n - u^*)|_{\partial\Omega_i}$.
3. Using the maximum principle and the equation (4.2) we obtain the estimation
$$\inf(u^i_n - u^*)|_{\partial\Omega_i} < (u^i_{n+1} - u^*)(x) < \sup(u^i_n - u^*)|_{\partial\Omega_i} \forall x \in \Omega_i \setminus \partial\Omega_i. \tag{4.3}$$
4. If we choose two different initial guesses such that
$$v(x) < (>) u(x) \quad \forall x \in \Omega \setminus \partial\Omega, \tag{4.4}$$
then in each step we have
$$v_n(x) < (>) u_n(x) \quad \forall x \in \Omega \setminus \partial\Omega. \tag{4.5}$$
Proof. By induction we have
$$v_n - u_n = \sum_{i=1}^{k} \chi_i (v^i_n - u^i_n),$$
but the right-hand side terms can be estimated by
$$v^i_n - u^i_n = R(\Omega_i, (v^i_{n-1} - u^i_{n-1})|_{\partial\Omega_i}, 0) < \sup(v_{n-1} - u_{n-1})|_{\partial\Omega_i} < 0.$$
Using further the maximum principle we get (4.5). The proof of the second inequality is analogous. ■

5. $$u_0(x) > u^*(x) \quad \forall x \in \Omega \setminus \partial\Omega \Rightarrow \forall n \quad u_n(x) > u^*(x) \quad \forall x \in \Omega \setminus \partial\Omega,$$
$$u_0(x) < u^*(x) \quad \forall x \in \Omega \setminus \partial\Omega \Rightarrow \forall n \quad u_n(x) < u^*(x) \quad \forall x \in \Omega \setminus \partial\Omega.$$
The proof follows from *3*, because using u^* as the initial element we get a constant sequence.
6. It is enough to find two initial guesses
$$u^+ > u^* > u^-, \tag{4.6}$$
such that
$$u^+_n \longrightarrow u^*, \qquad u^-_n \longrightarrow u^*, \tag{4.7}$$
then for every $u^+ > v_0 > u^-$, the sequence $v_n \longrightarrow u^*$.
The proof follows from *4*.

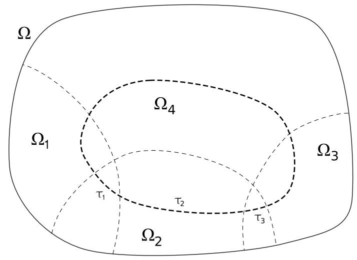

Figure 1. Example of non-regular overlapping

5. Proof of the Convergence

The partition into subdomains must fulfill some geometrical condition (see Fig. 1). The formal proof describes the necessary condition taking advantage of linearity of the PDE. It will be shown that the error is bounded by a geometric sequence and $\sup(|u_n - u^*|) \searrow 0$.

To describe these conditions, we need to introduce some notation:

- $\Gamma_i = \partial\Omega_i$
- $\Omega_i^0 = \Omega_i$, $\Omega_i^1 = \Omega \setminus \Omega_i$
- $\Lambda_j = \{\Gamma_j \cap (\bigcap_{i=1,\dots,k} \Omega_i^{s(i)}); \quad s \in \{0,1\}^k\}$
- $\Lambda = \bigcup \Lambda_i$

For each Ω_i define the sets

$$I(\Omega_i) = \{\tau \in \Lambda; \quad \tau \subset \Omega_i \quad \wedge \quad \partial\Omega_i \cap \tau = \emptyset\},$$
$$P(\Omega_i) = \{\tau \in \Lambda; \quad \tau \subset \Omega_i \quad \wedge \quad \partial\Omega_i \cap \tau \neq \emptyset\}.$$

We define recursively on Λ the subsets:

$$\begin{aligned}
B_0 &= \{\tau \in \Lambda; \quad \tau \subset \partial\Omega\},\\
O_0 &= \{\Omega_i; \quad \exists\tau \in B_0 \quad \tau \subset \partial\Omega_i\},\\
I_n &= \{\tau \in \Lambda; \quad \exists\Omega_j \in O_n \quad \tau \subset I(\Omega_j)\},\\
P_n &= \{\tau \in \Lambda; \quad \exists\Omega_j, \Omega_i \in O_n, i \neq j \quad \tau \subset P(\Omega_i) \cap P(\Omega_j),\}\\
B_{n+1} &= B_n \cup I_n \cup P_n,\\
O_{n+1} &= O_n \cup \{\Omega_i; \quad \exists\tau \in B_{n+1} \quad \tau \subset \partial\Omega_i\}.
\end{aligned}$$

Theorem 1. *If there is m such that $B_m = \Lambda$, then the Schwarz method converges and $q < 1$ exists, such that:*

$$\|u_{n+m} - u^*\|_{L^\infty} \leq q\|u_n - u^*\|_{L^\infty}. \tag{5.1}$$

Proof. *Step 1:* We show, that it is enough to estimate the error on the interfaces of the subdomains. Without loss of generality we can choose a initial guess which has a positive error in all Ω.

$$u^+ > u^*. \tag{5.2}$$

Then from observations 3 and 5 it follows that the error is positive in each step of the Schwarz method, and the subsolutions fulfill

$$0 < (u^i_{n+1} - u^*)(x) < \sup(u^i_n - u^*)|_{\partial\Omega_i} \quad \forall x \in \Omega_i \setminus \partial\Omega_i. \tag{5.3}$$

Then we can estimate the global error

$$0 < (u_{n+1} - u^*)(x) < \sum_{i=1}^{k}(\chi_i(x)\sup(u^i_n - u^*))|_{\partial\Omega_i} \quad \forall x \in \Omega \setminus \partial\Omega. \tag{5.4}$$

So, it is enough to show that

$$\sum_{i=1}^{k}(\chi_i(x)\sup(u^i_n - u^*))|_{\partial\Omega_j} \longrightarrow 0 \quad \forall j, \tag{5.5}$$

or equivalently

$$\max_j\{\sup(u_n - u^*)|_{\partial\Omega_j}\} \longrightarrow 0. \tag{5.6}$$

Step 2: Let $M = \max_j\{\sup(u_n - u^*)|_{\partial\Omega_j}\}$. We prove inductively by k, that

$$\sup_{\tau\in B_k}(u_n - u^*)|_\tau < M \Longrightarrow \sup_{\tau\in B_{k+1}}(u_{n+1} - u^*)|_\tau < M. \tag{5.7}$$

For B_0 we have $\sup_{\tau\in B_0}(u_n - u^*)|_\tau = 0\ \forall n$. Then we show for each subset that $B_{k+1} = B_k \cup I_k \cup P_k$. Note the following simple implications

$$\forall i = 0, \ldots, k \quad a_i < M \Longrightarrow \sum_{i=1}^{k}\chi_i(x)a_i < M, \tag{5.8}$$

$$\forall i = 0, \ldots, k \quad a_i \le M \wedge \exists j : a_j < M, \chi_j(x) > 0 \Longrightarrow \sum_{i=1}^{k}\chi_i(x)a_i < M. \tag{5.9}$$

If $\tau \in B_k$, then (5.7) follows from the inductive assumption. If $\tau \in I_k$, then it is inside of some subdomain Ω' where

$$\inf(u_n - u^*)|_{\partial\Omega'} < M. \tag{5.10}$$

It follows from the inductive assumption and definition of I_k, as well as from observation 3 we see that at least one of the subsolution on $\Omega' \setminus \partial\Omega'$ will be smaller that M. This is a example of the implication (5.8).

If $\tau \in P_k$, then there is a point x_0 where the error for some subdomain is $(u^i_{n+1} - u^*)(x_0) = M$. However, from the definition of P_k, it follows that the weight $\chi_i(x_0) < 1$ for some neighborhood of x_0 and there exists a second subdomain where $(u^j_{n+1} - u^*)(x_0) < M$, because $x_0 \in \Omega_j \setminus \partial\Omega_j$. This is a example of the implication (5.9). Thus the inductive step is finished.

The condition $B_m = \Lambda$ ensures that the error decreases after m steps of the Schwarz method.

Step 3: Estimation of the error by a geometric sequence.

To show Theorem 1 we choose the initial guess

$$\tilde{u}_0(x) = \begin{cases} u^*(x) + M & \text{if } x \in \Omega \setminus \partial\Omega, \\ u^*(x) & \text{if } x \in \partial\Omega. \end{cases} \tag{5.11}$$

and by the monotony (4.5) of the sequence we can estimate

$$\|u_m - u^*\|_{L^\infty} \leq \|\tilde{u}_m - u^*\|_{L^\infty} \leq q\|\tilde{u}_0 - u^*\|_{L^\infty} = q\|u_0 - u^*\|_{L^\infty}, \quad q < 1. \tag{5.12}$$

This estimation does not depends on M, because the operator L is linear so

$$\alpha(\tilde{u}_m) = (\alpha\tilde{u}_0)_m. \tag{5.13}$$

■

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